# Diameter of $G_{np}$ and the Small-World Phenomenon

CS224W: Analysis of Networks
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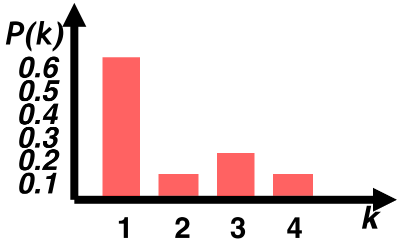
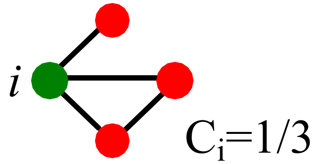
http://cs224w.stanford.edu

# Recap: Network Properties & $G_{np}$

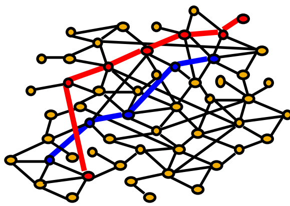
## How to characterize networks?




- Connectivity
- Degree distribution *P(k)*




- Clustering Coefficient *C*
- Diameter (shortest path length) *h*



## How to model networks?

- **Erdös-Renyi Random Graph** [Erdös-Renyi, '60]
  - $G_{n,p}$: undirected graph on $n$ nodes where each edge $(u,v)$ appears independently with prob. $p$

# Random Graph Model: Edges

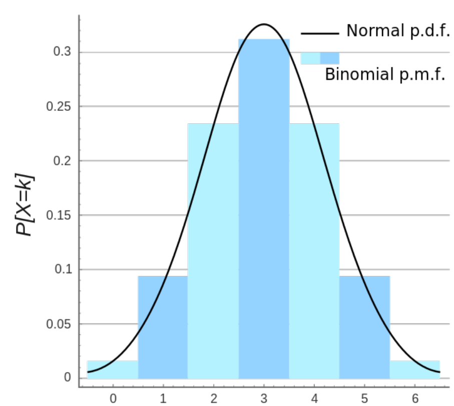
- **How likely is a graph on $E$ edges?**
- $P(E)$: the probability that a given $G_{np}$ generates a graph on exactly $E$ edges:

$$P(E) = \binom{E_{\max}}{E} p^{E} (1-p)^{E_{\max}-E}$$

where $E_{max}=n(n-1)/2$ is the maximum possible number of edges in an undirected graph of $n$ nodes

**P(E) is exactly the**
**Binomial distribution** **>>>**

Number of successes in a sequence of $E_{max}$ independent yes/no experiments



# Node Degrees in a Random Graph

## What is expected degree of a node?

- Let $X_v$ be a rnd. var. measuring the degree of node $v$
- **We want to know $E[X_v]$:**

**Approach:**

- Decompose $X_v$ to $X_v = X_{v,1}+X_{v,2}+\ldots+X_{v,n-1}$
  - where $X_{v,u}$ is a $\{0,1\}$-random var. which tells if edge $(v,u)$ exists or not
  - **For the calculation we will need: Linearity of expectation**
    - For any random variables $Y_1, Y_2, \ldots, Y_k$
    - If $Y=Y_1+Y_2+\ldots Y_k$, then $E[Y]= \sum_i E[Y_i]$

$$E[X_v] = \sum_{u=1}^{n-1} E[X_{vu}] = (n-1)p$$

**How to think about this?**

- Prob. of node $u$ linking to node $v$ is $p$
- $u$ can link (flips a coin) to all other $(n-1)$ nodes
- Thus, the expected degree of node $u$ is: $p(n-1)$

# Properties of $G_{np}$

**Degree distribution:** $P(k)$

**Path length:** $h$

**Clustering coefficient:** $C$

**What are the values of these properties for $G_{np}$?**

# Degree Distribution

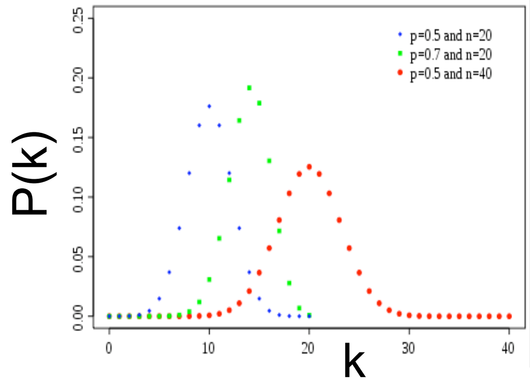
- **Fact:** Degree distribution of $G_{np}$ is binomial.
- Let $P(k)$ denote the fraction of nodes with degree $k$:

$$P(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

Select $k$ nodes out of $n$-$1$

Probability of having $k$ edges

Probability of missing the rest of the $n$-$1$-$k$ edges




**Mean, variance of a binomial distribution**

$$\bar{k} = p(n-1)$$

$$\sigma^2 = p(1-p)(n-1)$$

$$\frac{\sigma}{\bar{k}} = \left[\frac{1-p}{p}\frac{1}{(n-1)}\right]^{1/2} \approx \frac{1}{(n-1)^{1/2}}$$

By the law of large numbers, as the network size increases, the distribution becomes increasingly narrow—we are increasingly confident that the degree of a node is in the vicinity of $k$.

# Clustering Coefficient of $G_{np}$

- **Remember:** $C_i = \frac{2e_i}{k_i(k_i-1)}$

  Where $e_i$ is the number of edges between i's neighbors

- Edges in $G_{np}$ appear i.i.d. with prob. $p$

- **So:** $e_i = p\frac{k_i(k_i-1)}{2}$

  Each pair is connected with prob. $p$

  Number of distinct pairs of neighbors of node $i$ of degree $k_i$

- **Then (expected):** $C = \frac{p \cdot k_i(k_i-1)}{k_i(k_i-1)} = p = \frac{\bar{k}}{n-1} \approx \frac{\bar{k}}{n}$

Clustering coefficient of a random graph is small.
If we generate bigger and bigger graphs with fixed avg. degree $k$ (that is we set $p = k \cdot 1/n$), then $C$ decreases with the graph size $n$.

# Network Properties of $G_{np}$

**Degree distribution:** $P(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

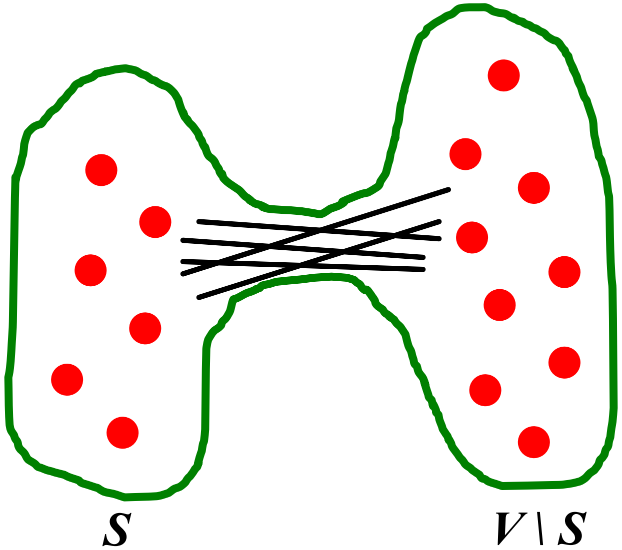
**Clustering coefficient:** $C = p = \bar{k}/n$

**Path length:** ***next!***

# Two Definitions

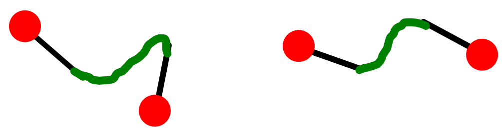
- **We need to define two concepts**
- **1) Define: Random k-Regular graph**
  - Assume each node has $k$ spokes (half-edges)
  - Randomly pair them up!

- **2) Define: Expansion**
  - Graph $G(V, E)$ has **expansion $\alpha$**: if $\forall S \subseteq V$: #edges leaving $S$ $\geq \alpha \cdot \min(|S|, |V \setminus S|)$



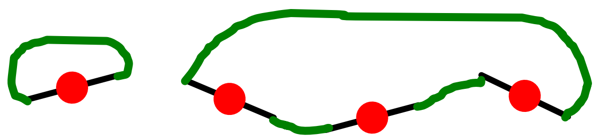
# Def 1 : Random k-Regular Graphs

- To prove the diameter of a $G_{np}$ we define a few concepts
- **Define: Random k-Regular graph**
  - **Assume each node has $k$ spokes (half-edges)**
    - **k=1:**



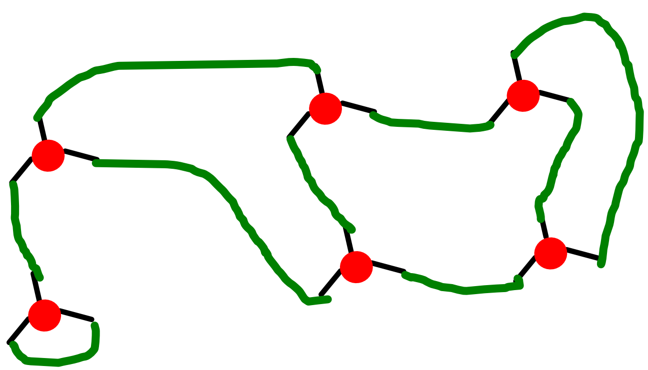
**Graph is a set of pairs**

    - **k=2:**



**Graph is a set of cycles**

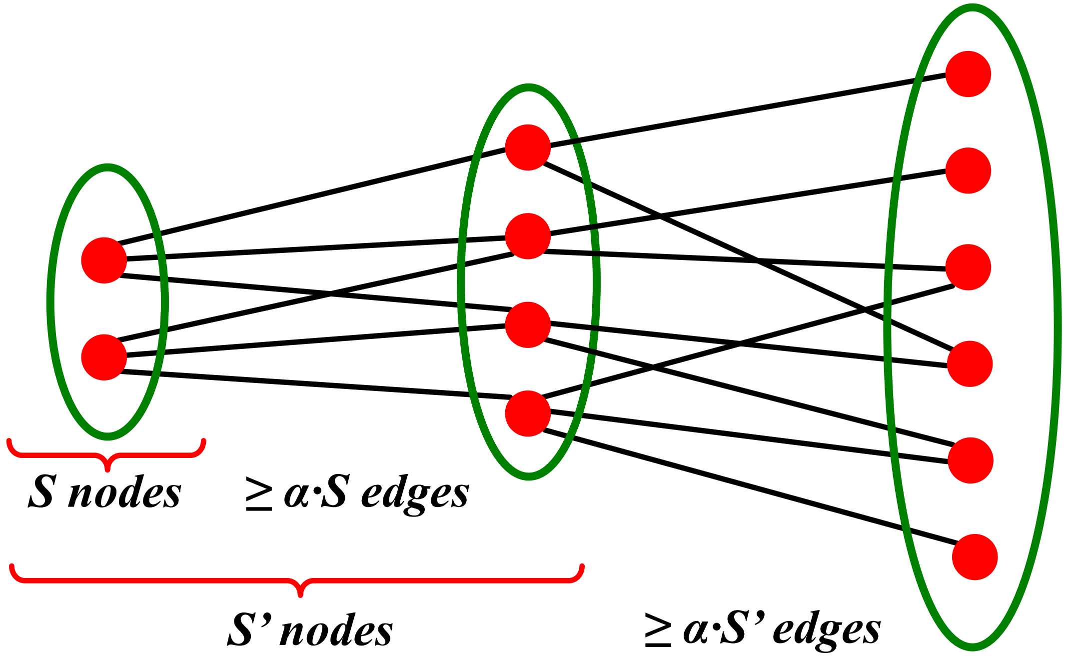
    - **k=3:**



**Arbitrarily complicated graphs**

  - **Randomly pair them up!**

# Def 2: Expansion: Intuition



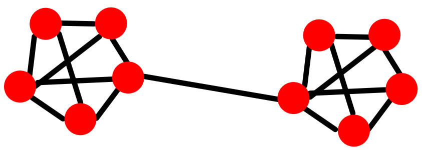

$$\alpha = \min_{S \subseteq V} \frac{\#\, edges\ leaving\ S}{\min(|S|, |V \setminus S|)}$$

**(A big) graph with "good" expansion**

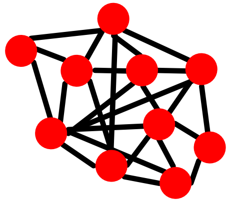
# Expansion: Measures Robustness

$$\alpha = \min_{S \subseteq V} \frac{\#\,edges\ leaving\ S}{\min(|S|, |V \setminus S|)}$$

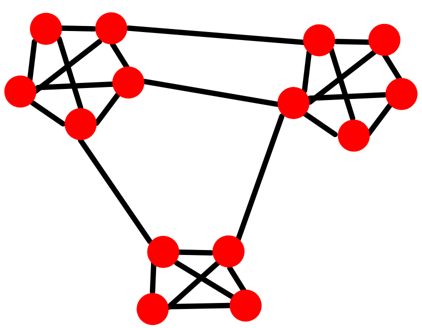
- Expansion is **measure of robustness:**
  - To disconnect $l$ nodes, we need to cut $\geq \alpha \cdot l$ edges
- Low expansion:



- High expansion:



- Social networks:
  - "Communities"



# Def 2: Expansion of k-Regular Graphs

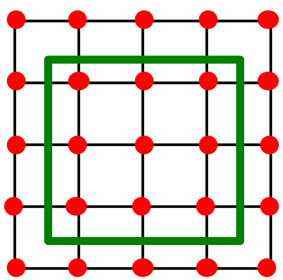
$$\alpha = \min_{S \subseteq V} \frac{\#\, edges\ leaving\ S}{\min(|S|, |V \setminus S|)}$$

- ***k*-regular graph**:
  - Expansion is at most $k$ (when $S$ is a single node)

- **Is there a graph on $n$ nodes ($n \to \infty$), of fixed max deg. $k$, so that expansion $\alpha$ remains const?**

**Examples:**

- **n×n grid:** $k=4$: $\alpha = 2n/(n^2/4) \to 0$
  ($S = n/2 \times n/2$ square in the center)

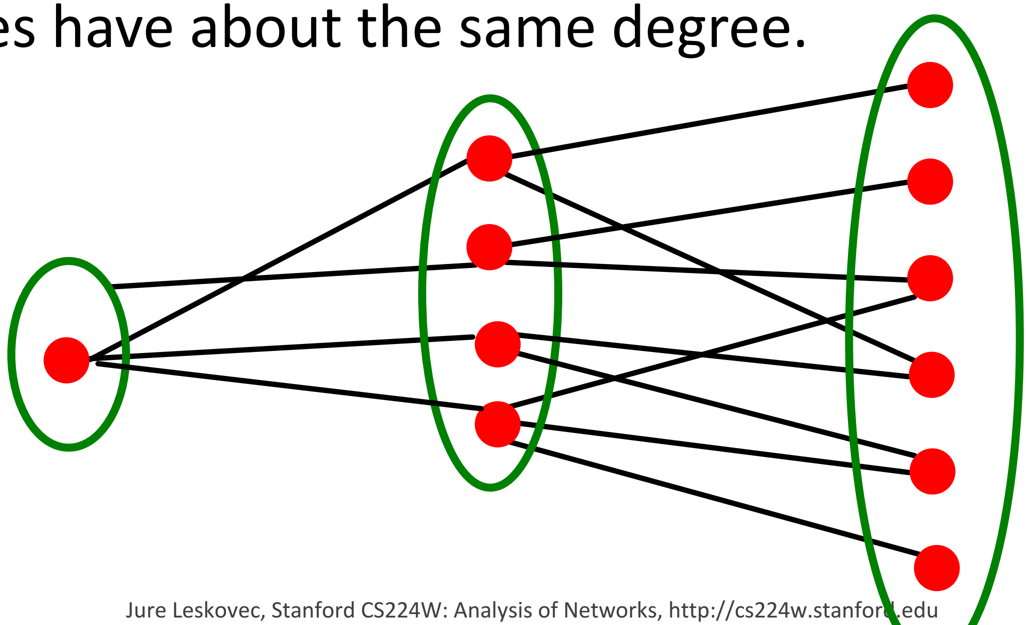



- **Fact:** For a random **3-regular graph** on $n$ nodes, there is some const $\alpha$ ($\alpha > 0$, independent. of $n$) such that with high prob. (prob→1 as n→∞) the expansion of the graph is $\geq \alpha$.

In fact, $\alpha = \Theta(k/2)$ as $n \to \infty$

# Diameter of k-Regular Rnd. Graph

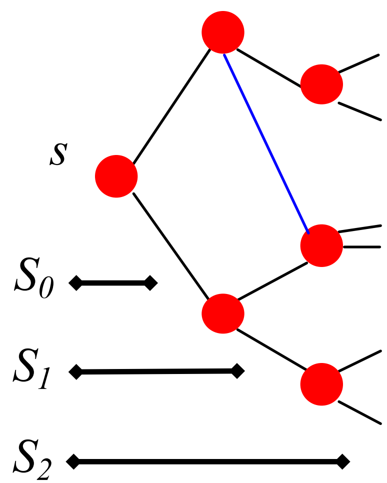
- **We will prove the diameter of a *k*-regular random graph (k=3)**
  - Note that *k*-regular random graph are essentially the same as random graphs
    - In a random graph variance/mean degree goes to 0 as graph size increases, which means that intuitively all nodes have about the same degree.



# Diameter of k-Regular Rnd. Graph

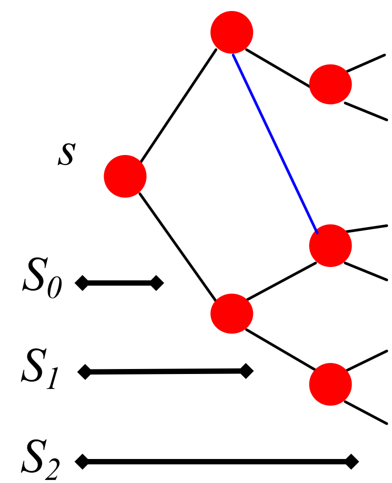
**Fact:** In a $k$-regular random graph on $n$ nodes with expansion $\alpha$, for all pairs of nodes $s$ and $t$ there is a path of $O((\log n) / \alpha)$ edges connecting them.

- Proof:
  - Proof strategy:
    - We want to show that from any node $s$ there is a path of length $O((\log n)/\alpha)$ to any other node $t$
  - Let $S_j$ be a set of all nodes found within $j$ steps of BFS from $s$.
  - **How does $S_j$ increase as a function of $j$?**

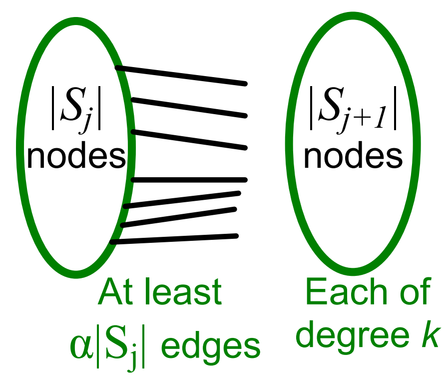


# Diameter of k-Regular Rnd. Graph

- <u>Proof (continued):</u>
  - Let $S_j$ be a set of all nodes found within $j$ steps of BFS from $s$.
  - **We want to relate $S_j$ and $S_{j+1}$**




$$|S_{j+1}| \geq |S_j| + \frac{\overbrace{\alpha|S_j|}^{\text{Expansion}}}{\underbrace{k}_{\substack{\text{At most } k \text{ edges} \\ \text{“collide” at a node}}}} =$$




$$|S_{j+1}| \geq |S_j| \left(1 + \frac{\alpha}{k}\right) = S_0 \left(1 + \frac{\alpha}{k}\right)^{j+1}$$

where $S_0 = 1$

# Diameter of k-Regular Rnd. Graph

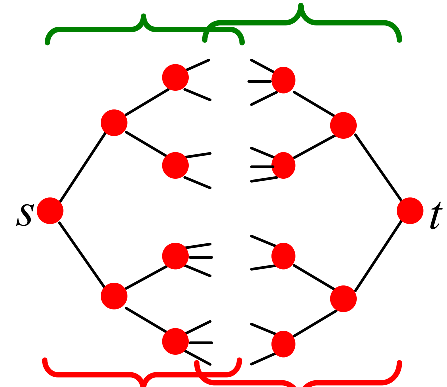
$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

- Proof (continued):
  - **In how many steps of BFS do we reach $>n/2$ nodes?**
  - Need $j$ so that: $S_j = \left(1+\frac{\alpha}{k}\right)^j \geq \frac{n}{2}$
  - Let's set: $j = \frac{k \log_2 n}{\alpha}$
  - Then:

$$\left(1+\frac{\alpha}{k}\right)^{\frac{k\log_2 n}{\alpha}} \geq 2^{\log_2 n} = n > \frac{n}{2}$$

  - In $2k/\alpha \cdot \log n$ steps $|S_j|$ grows to $\Theta(n)$. So, **the diameter of $G$ is $O(\log(n)/\alpha)$**

In $j$ steps, we reach $>n/2$ nodes | In $j$ steps, we reach $>n/2$ nodes | $\Rightarrow$ Diameter = $2\cdot j$




In $log(n)$ steps, we reach $>n/2$ nodes | In $log(n)$ steps, we reach $>n/2$ nodes | $\Rightarrow$ **Diameter = $2\log(n)$**

**Claim:**

$$\left(1+\frac{\alpha}{k}\right)^{\frac{k\log_2 n}{\alpha}} \geq 2^{\log_2 n}$$

**Remember $n>0$, $\alpha \leq k$ then:**

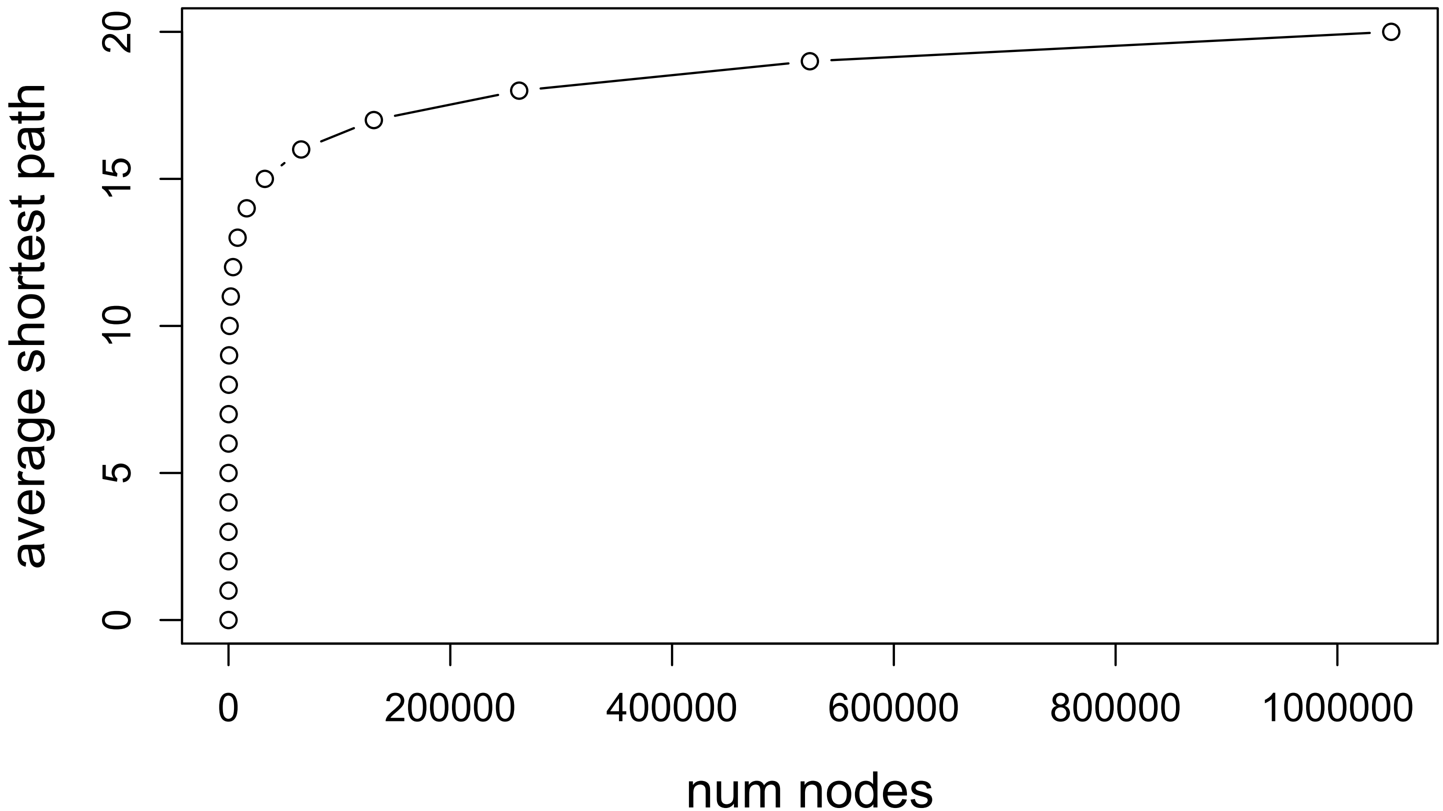
$$\text{if } \alpha = k: (1+1)^{\frac{1}{1}\log_2 n} = 2^{\log_2 n}$$

$$\text{if } \alpha \to 0 \text{ then } \frac{k}{\alpha} = x \to \infty:$$

$$\text{and } \left(1+\frac{1}{x}\right)^{x\log_2 n} = e^{\log_2 n} > 2^{\log_2 n}$$

# Erdös-Renyi avg. shortest path

- Erdös-Renyi networks can grow to be very large but nodes will be just a few hops apart



# Back to Node Degrees of $G_{np}$

Extra

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

- **Remember, expected degree:** $E[X_v] = (n-1)p$
- **To have const. degree we want $E[X_v]$ be independent of $n$:** So let: $p=k/(n-1)$
- **Observation:** If we build random graph $G_{np}$ with $p=k/(n-1)$ we have many isolated nodes
- **Why?**

$$P[v \text{ has degree } 0] = (1-p)^{n-1} = \left(1-\frac{k}{n-1}\right)^{n-1} \xrightarrow[n\to\infty]{} e^{-k}$$

**Why?**

$$\lim_{n\to\infty}\left(1-\frac{k}{n-1}\right)^{n-1} = \left(1-\frac{1}{x}\right)^{x\cdot k} = \left[\underbrace{\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{-x}}_{e}\right]^{-k} = e^{-k}$$

Use substitution $\frac{1}{x} = \frac{k}{n-1}$

**In other words:** In the limit of $n\to\infty$ Poisson distribution is a good approximation of a Binomial distribution and we computed $P(x=0)$ for the Poisson PMF.

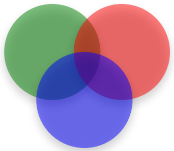
# No Isolated Nodes

**Extra**

- **How big do we have to make $p$ before we are likely to have no isolated nodes?**
- We know: $\boldsymbol{P[v \text{ has degree } 0] = e^{-k}}$
- **Event we are asking about is:**
  - $I$ = some node is isolated
  - $I = \bigcup_{v \in N} I_v$ where $\boldsymbol{I_v}$ is the event that $\boldsymbol{v}$ is isolated
- **We have:**

$$P(I) = P\left(\bigcup_{v \in N} I_v\right) \le \sum_{v \in N} P(I_v) = ne^{-k}$$

Union bound



$A_i$

$$\left|\bigcup_i A_i\right| \le \sum_i |A_i|$$

# No Isolated Nodes

Extra

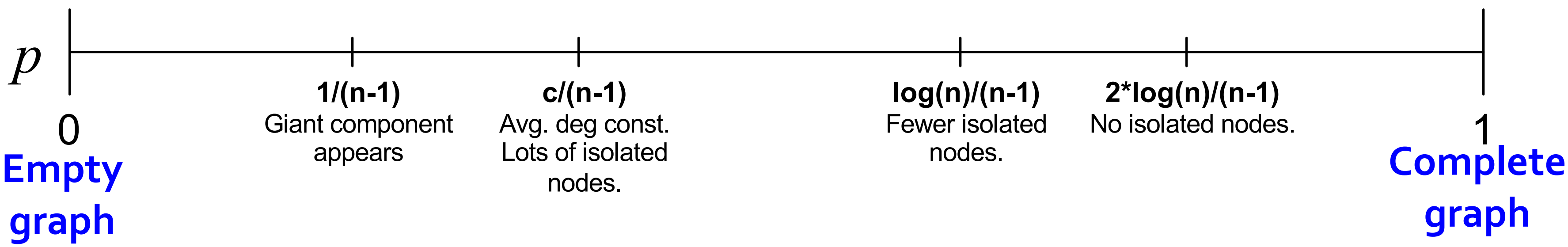
- **We just learned: $P(I) \leq n\, e^{-k}$**
- **Let's try:**
  - $k = \ln n$ then: $n\, e^{-k} = n\, e^{-\ln n} = n \cdot 1/n = 1$
  - $k = 2 \ln n$ then: $n\, e^{-2 \ln n} = n \cdot 1/n^2 = 1/n$
- **So if:**
  - $k = \ln n$ then: $P(I) \leq 1$
  - $k = 2 \ln n$ then: $P(I) \leq 1/n \to 0$ as $n \to \infty$

So, for ***p=2ln(n)*** we get no isolated nodes (as $n \to \infty$)

# "Evolution" of a Random Graph

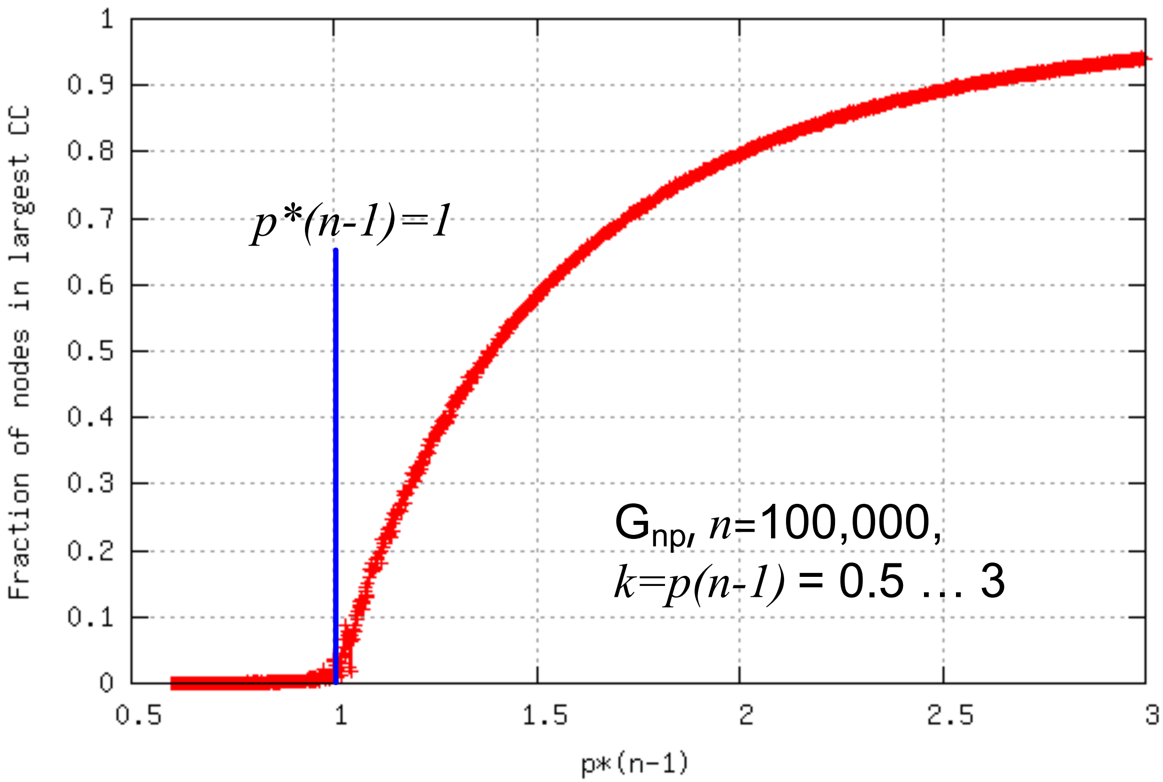
Extra

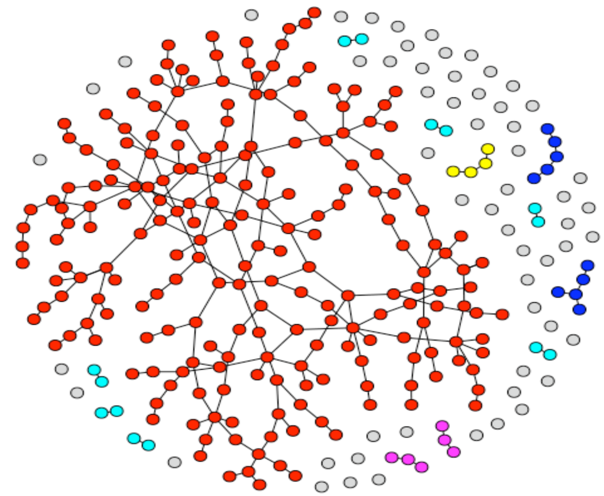
- **Graph structure of $G_{np}$ as $p$ changes:**




- **Emergence of a giant component:**
  avg. degree $k=2E/n$ or $p=k/(n-1)$
  - $k=1-\varepsilon$: all components are of size $\Omega(\log n)$
  - $k=1+\varepsilon$: 1 component of size $\Omega(n)$, others have size $\Omega(\log n)$

# $G_{np}$ Simulation Experiment



Fraction of nodes in the largest component

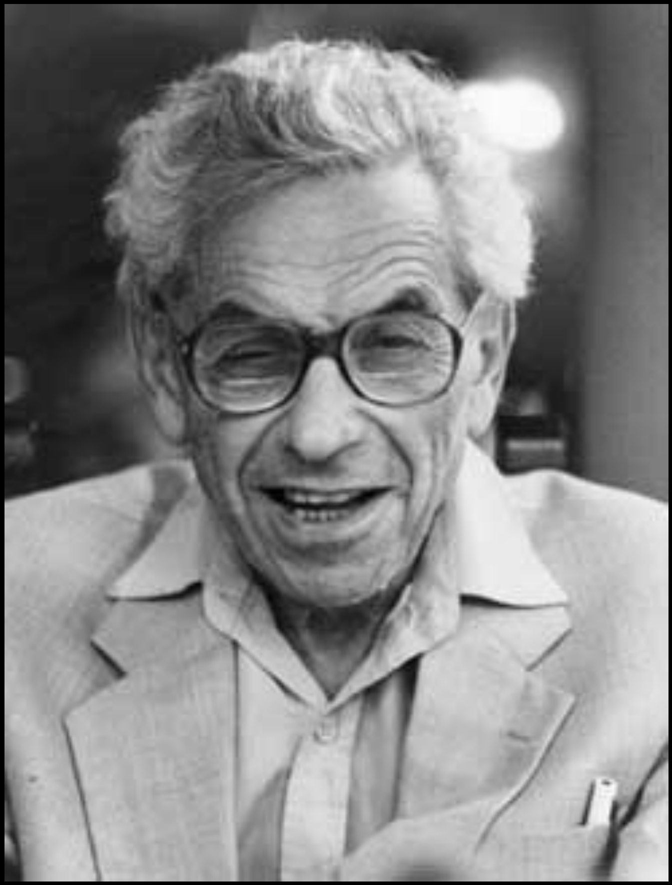
- **Emergence of a giant component:**
  avg. degree $k=2E/n$ or $p=k/(n-1)$
  - $k=1-\varepsilon$: all components are of size $\Omega(\log n)$
  - $k=1+\varepsilon$: 1 component of size $\Omega(n)$, others have size $\Omega(\log n)$

# Network Properties of $G_{np}$

**Degree distribution:** $P(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

**Path length:** $O(\log n)$

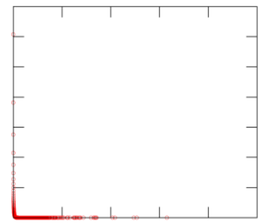
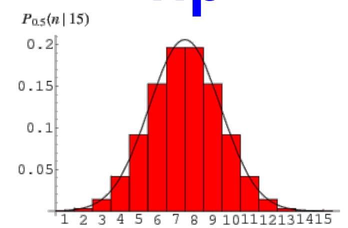
**Clustering coefficient:** $C = p = \bar{k} / n$


Paul Erdös

# $G_{np}$ is so cool!
# Let's compare it to real networks.


DIRECT FROM
RINGSIDE!
THE FIGHT EVERYONE WANTS TO SEE...
MSN
BOMBER
VS
$G_{np}$
ROCKER
ROCK'EM
SOCK'EM
ROBOTS
MAIN EVENT

# Back to MSN vs. $G_{np}$

| | MSN | $G_{np}$ | n=180M |
|---|---|---|---|
| **Degree distribution:** |  |   | |
| **Path length:** | **6.6** | $O(\log n)$<br>$h \approx 8.2$ | |
| **Clustering coefficient:** | $0.11$ | $\bar{k}/n$<br>$C \approx 8 \cdot 10^{-8}$ | |
| **Connected component:** | $99\%$ | GCC exists when $\bar{k}>1$.<br>$\bar{k} \approx 14$. | |

# Real Networks vs. $G_{np}$

- Are real networks like random graphs?
  - Average path length: ☺
  - Giant connected component: ☺
  - Clustering Coefficient: ☹
  - Degree Distribution: ☹
- Problems with the random network model:
  - Degree distribution differs from that of real networks
  - Giant component in most real networks
    does NOT emerge through a phase transition
  - No "local" structure – clustering coefficient is too low
- Most important: Are real networks random?
  - The answer is simply: **NO!**

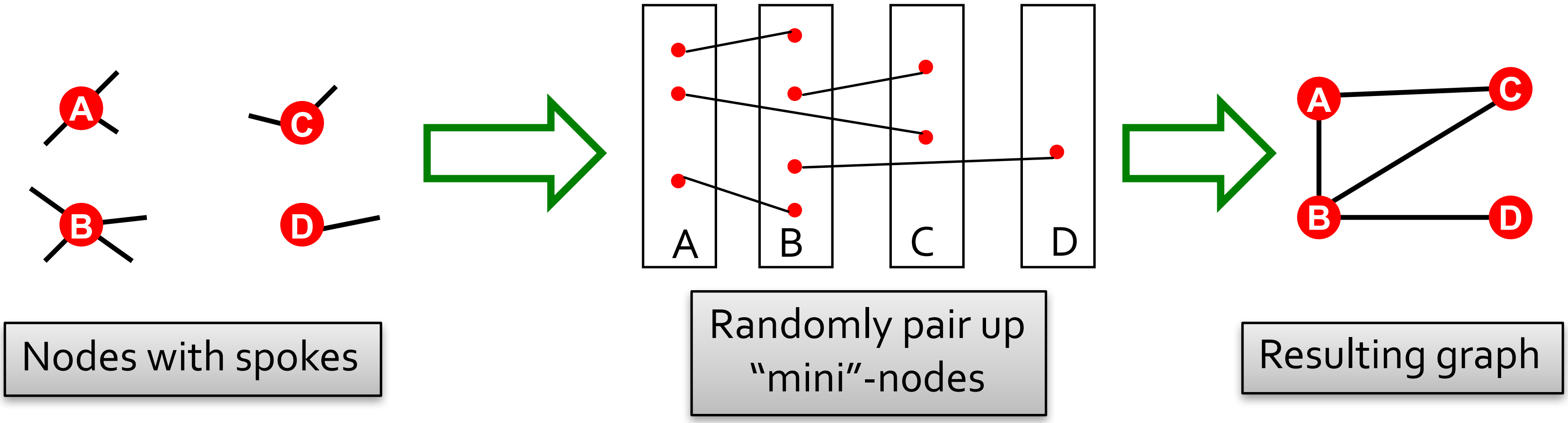
# Real Networks vs. $G_{np}$

- **If $G_{np}$ is wrong, why did we spend time on it?**
  - It is the reference model for the rest of the class
  - It will help us calculate many quantities, that can then be compared to the real data
  - It will help us understand to what degree is a particular property the result of some random process

**So, while $G_{np}$ is WRONG, it will turn out to be extremly USEFUL NULL MODEL!**

# Intermezzo: Configuration Model

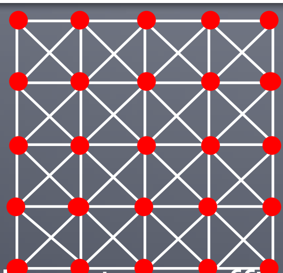
- **Goal:** Generate a random graph with a given degree sequence $k_1, k_2, \ldots k_N$
- **Configuration model:**




- **Useful as a "null" model of networks:**
  - We can compare the real network $G$ and a "random" $G'$ which has the same degree sequence as $G$

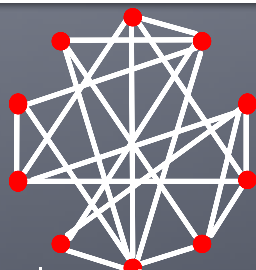
# The Small-World Model

Can we have high clustering while also having short paths?



High clustering coefficient,
High diameter

Vs.



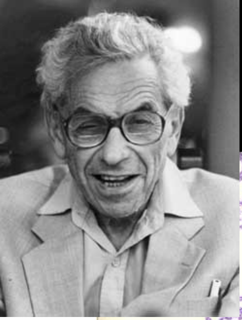
Low clustering coefficient
Low diameter

# Six Degrees of Kevin Bacon

## Origins of a small-world idea:

- **The Bacon number:**
  - Create a network of Hollywood actors
  - Connect two actors if they co-appeared in the movie
  - **Bacon number:** number of steps to Kevin Bacon
- As of Dec 2007, the highest (finite) Bacon number reported is 8
- Only approx. 12% of all actors cannot be linked to Bacon


Erdös numbers are small!
(Jure's number is 3.)



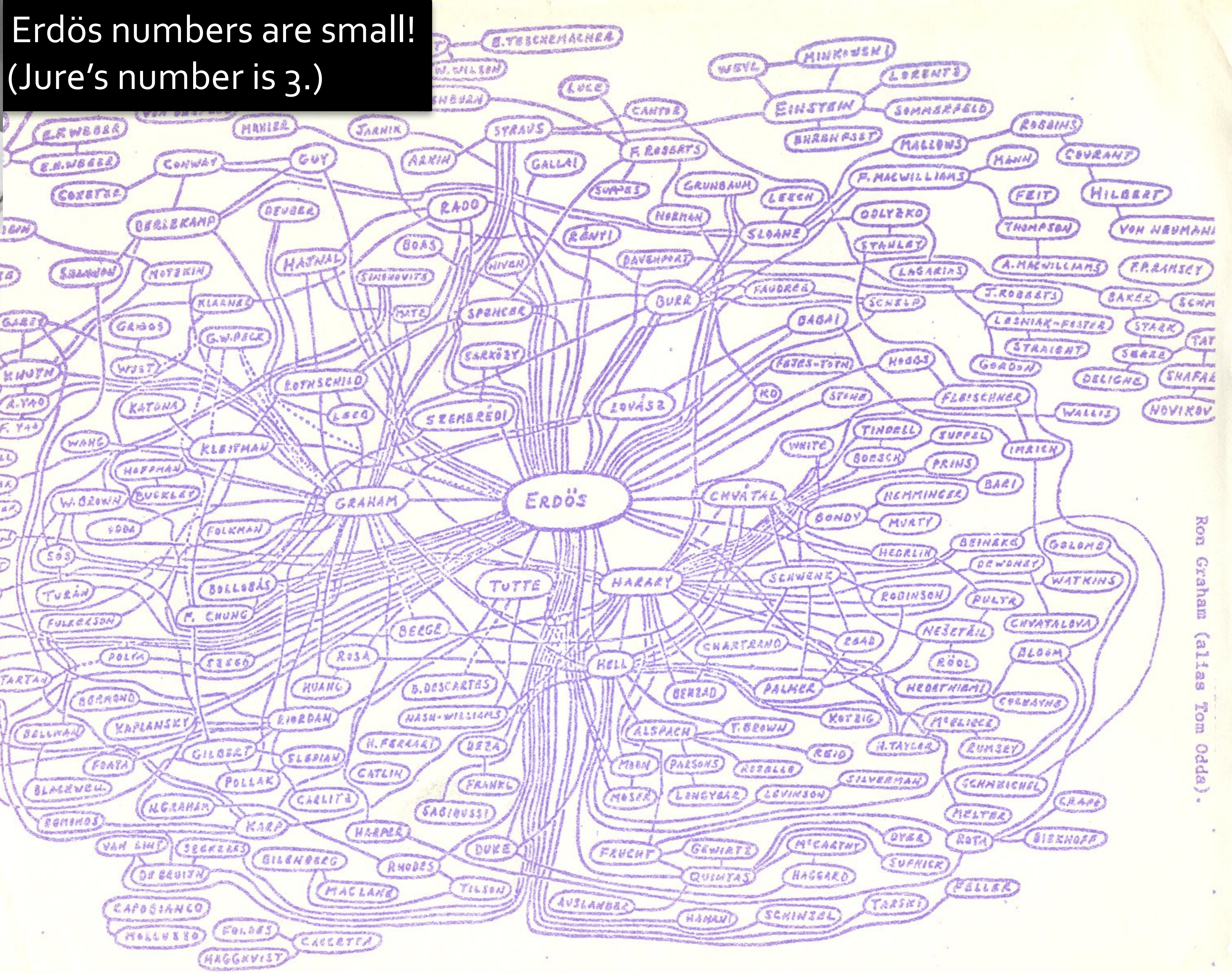


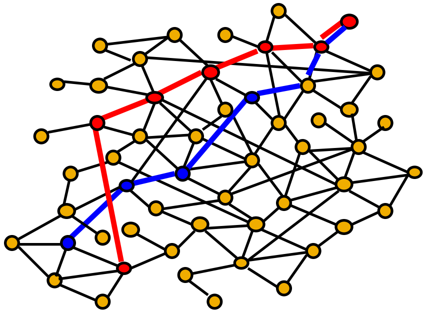

Figure 1
To appear in Topics in Graph Theory (F. Harary, ed.) New York Academy of Sciences (1979).

Find out your Erdos number: http://www.ams.org/mathscinet/collaborationDistance.html

# Time for a joke (via XKCD): What do you do when dead start walking the earth?


# The Small-World Experiment

- **What is the typical shortest path length between any two people?**
  - **Experiment on the global friendship network**
    - Can't measure, need to probe explicitly
- **Small-world experiment** [Milgram '67]
  - Picked 300 people in Omaha, Nebraska and Wichita, Kansas
  - Ask them to get a letter to a stock-broker in Boston by passing it through friends
- **How many steps did it take?**

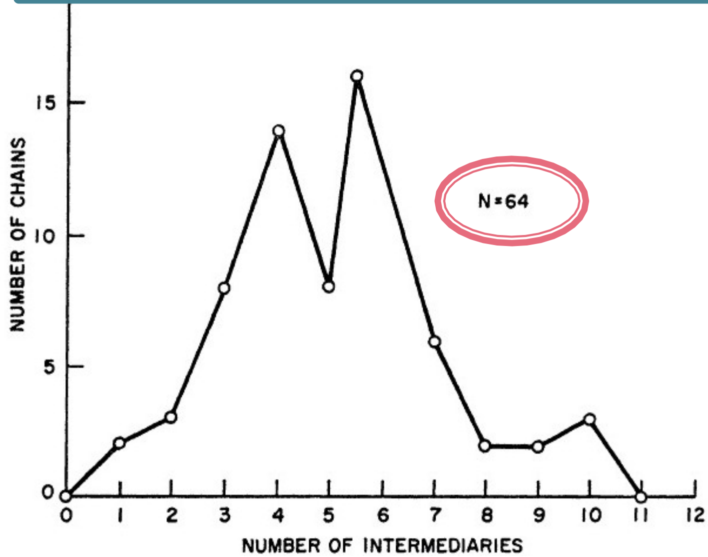


# The Small-World Experiment

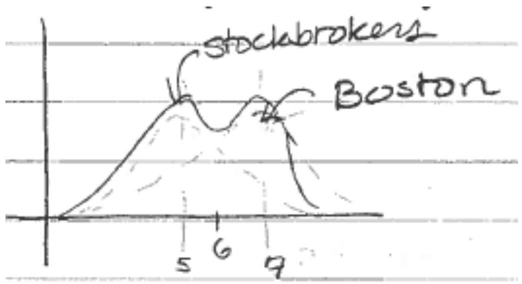
[Milgram, '67]

- **64 chains completed:**
  (i.e., 64 letters reached the target)
  - It took 6.2 steps on the average, thus **"6 degrees of separation"**
- **Further observations:**
  - People who owned stock had shorter paths to the stockbroker than random people: 5.4 vs. 6.7
  - People from the Boston area have even closer paths: 4.4

Milgram's small world experiment



# Milgram: Further Observations

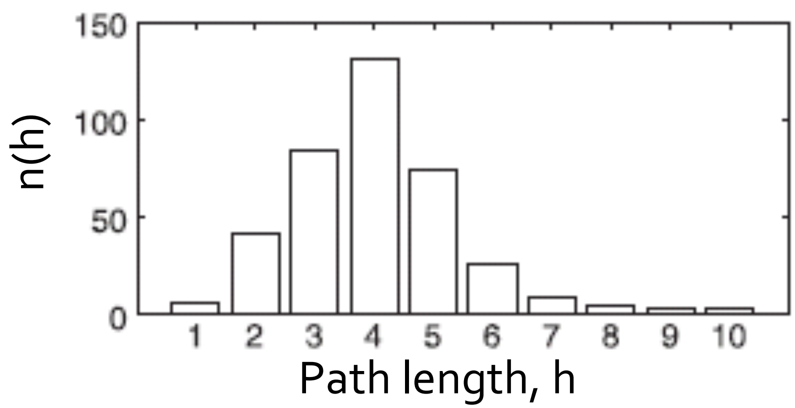



- **Boston vs. occupation networks:**
- **Criticism:**
  - **Funneling:**
    - 31 of 64 chains passed through 1 of 3 people as their final step → Not all links/nodes are equal
  - Starting points and the target were non-random
  - There are not many samples (only 64)
  - People refused to participate (25% for Milgram)
    - Not all searches finished (only 64 out of 300)
  - **Some sort of social search:** People in the experiment follow some strategy instead of forwarding the letter to everyone. **They are not finding the shortest path!**
  - People might have used extra information resources

[Dodds-Muhamad-Watts, '03]

# Columbia Small-World Study

- **In 2003 Dodds, Muhamad and Watts performed the experiment using e-mail:**
  - 18 targets of various backgrounds
  - 24,000 first steps (~1,500 per target)
  - 65% dropout per step
  - 384 chains completed (1.5%)




**Avg. chain length = 4.01**

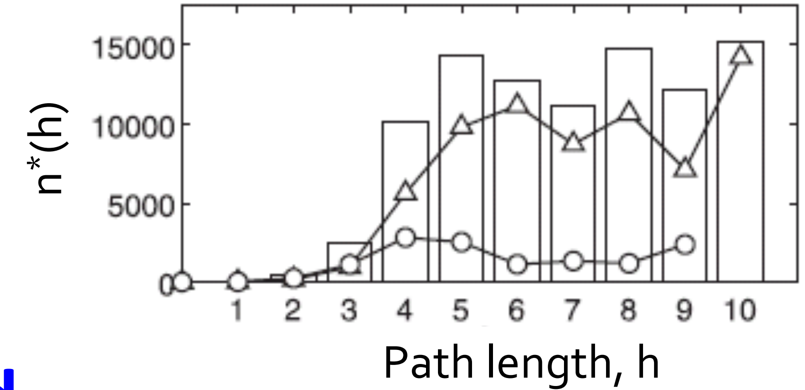
**Problem:** People stop participating

Correction factor: $$n^*(h) = \frac{n(h)}{\prod_{i=0}^{h-1}(1-r_i)}$$

$r_i$ .... drop-out rate at hop $i$

# Small-World in Email Study

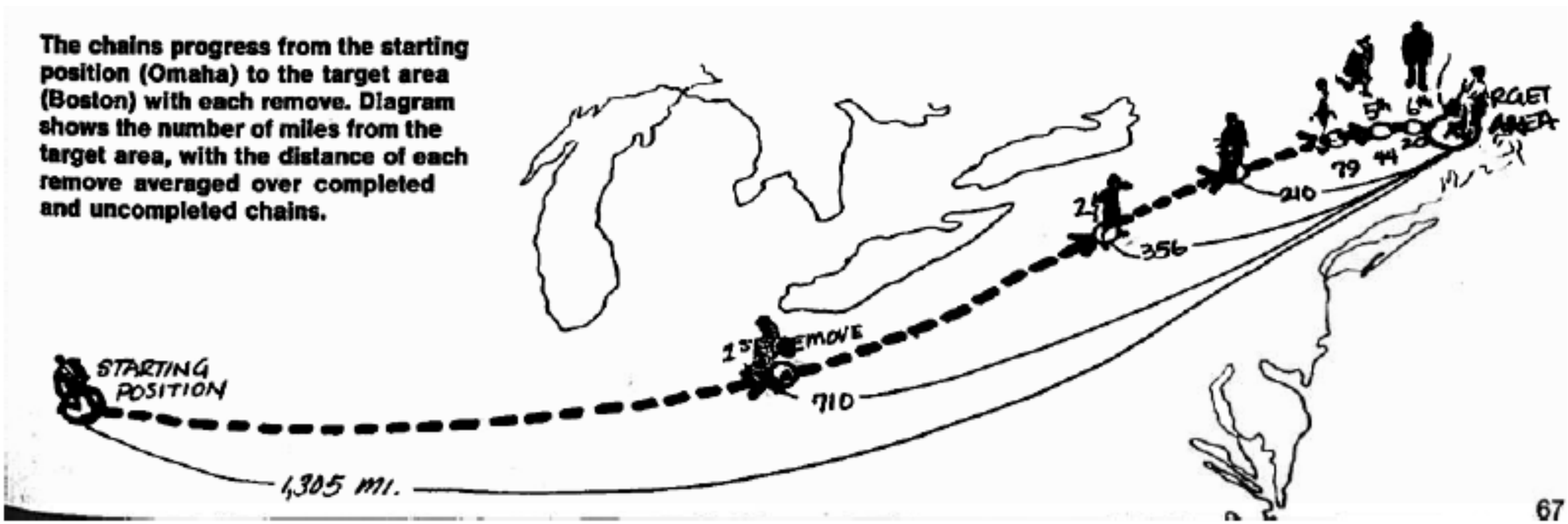
- **After the correction:**
  - **Typical path length h = 7**




- **Some not well understood phenomena in social networks:**
  - **Funneling effect:** Some target's friends are more likely to be the final step
    - Conjecture: High reputation/authority
  - **Effects of target's characteristics:** Structurally, why are high-status targets easier to find?
    - Conjecture: Core-periphery network structure

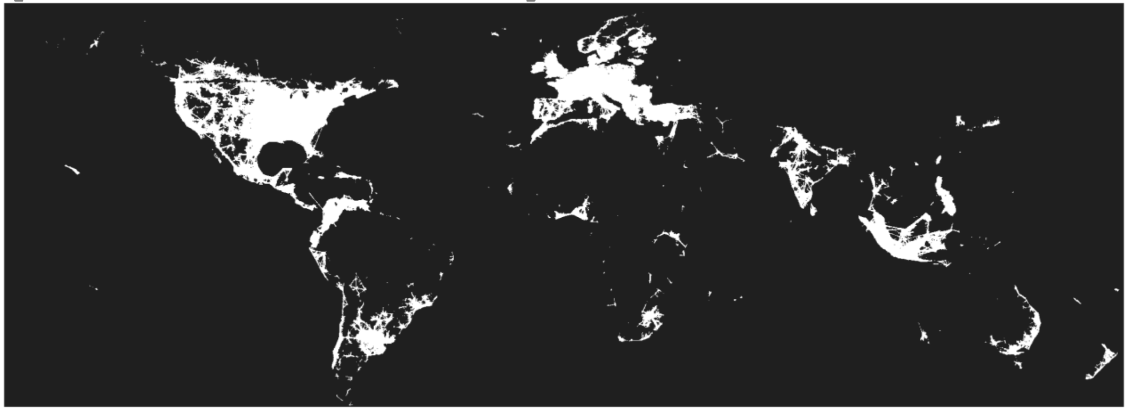
# Two Questions

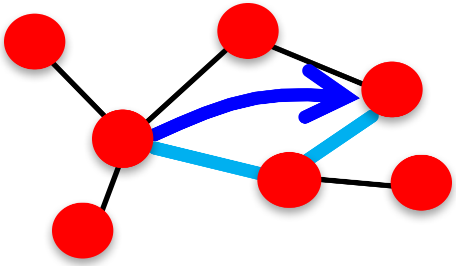
- (Today) What is the structure of a social network?
- (Next class) What kind of mechanisms do people use to route and find the target?



# 6-Degrees: Should We Be Surprised?

- **Assume each human is connected to 100 other people**
  **Then:**
  - Step 1: reach 100 people
  - Step 2: reach 100*100 = 10,000 people
  - Step 3: reach 100*100*100 = 1,000,000 people
  - Step 4: reach 100*100*100*100 = 100M people
  - **In 5 steps we can reach 10 billion people**
- **What's wrong here?**
  - **92% of new FB friendships are to a friend-of-a-friend** [Backstom-Leskovec '11]





# Clustering Implies Edge Locality

- **MSN network has 7 orders of magnitude larger clustering than the corresponding $G_{np}$!**
- **Other examples:**

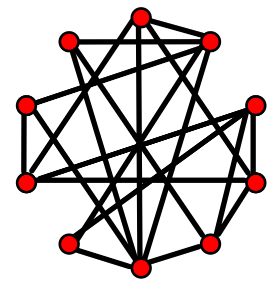
Actor Collaborations (IMDB): $N$ = 225,226 nodes, avg. degree $\overline{k}$ = 61
Electrical power grid: $N$ = 4,941 nodes, $\overline{k}$ = 2.67
Network of neurons: $N$ = 282 nodes, $\overline{k}$ = 14

| Network | $h_{actual}$ | $h_{random}$ | $C_{actual}$ | $C_{random}$ |
|---|---|---|---|---|
| **Film actors** | 3.65 | 2.99 | 0.79 | 0.00027 |
| **Power Grid** | 18.70 | 12.40 | 0.080 | 0.005 |
| **C. elegans** | 2.65 | 2.25 | 0.28 | 0.05 |

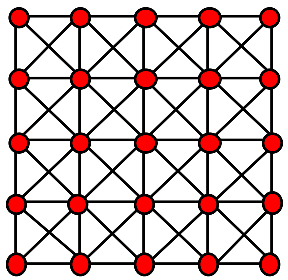
h ... Average shortest path length
C ... Average clustering coefficient
"actual" … real network
"random" … random graph with same avg. degree

# The “Controversy”

- **Consequence of expansion:**
  - **Short paths: $O(\log n)$**
    - This is the smallest diameter we can get if we have a constant degree.
  - But clustering is low!
- **But networks have “local” structure:**
  - **Triadic closure:** Friend of a friend is my friend
  - High clustering but diameter is also high
- **How can we have both?**



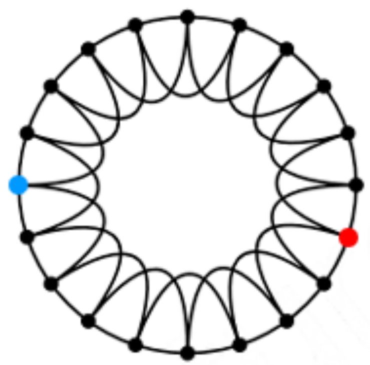
**Low diameter**
**Low clustering coefficient**



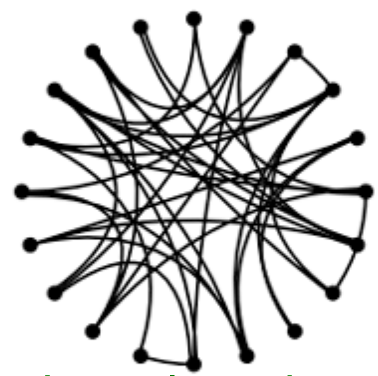
**High clustering coefficient**
**High diameter**

# Small-World: How?

- **Could a network with high clustering be at the same time a small world?**
  - How can we at the same time have **high clustering** and **small diameter?**



High clustering
High diameter



Low clustering
Low diameter

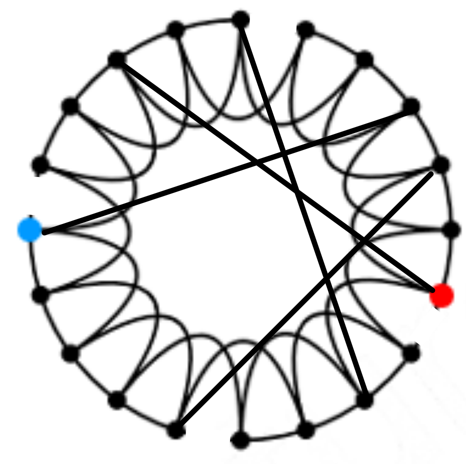
  - Clustering implies edge “locality”
  - Randomness enables “shortcuts”

# Solution: The Small-World Model

**Small-World Model** **[Watts-Strogatz '98]**

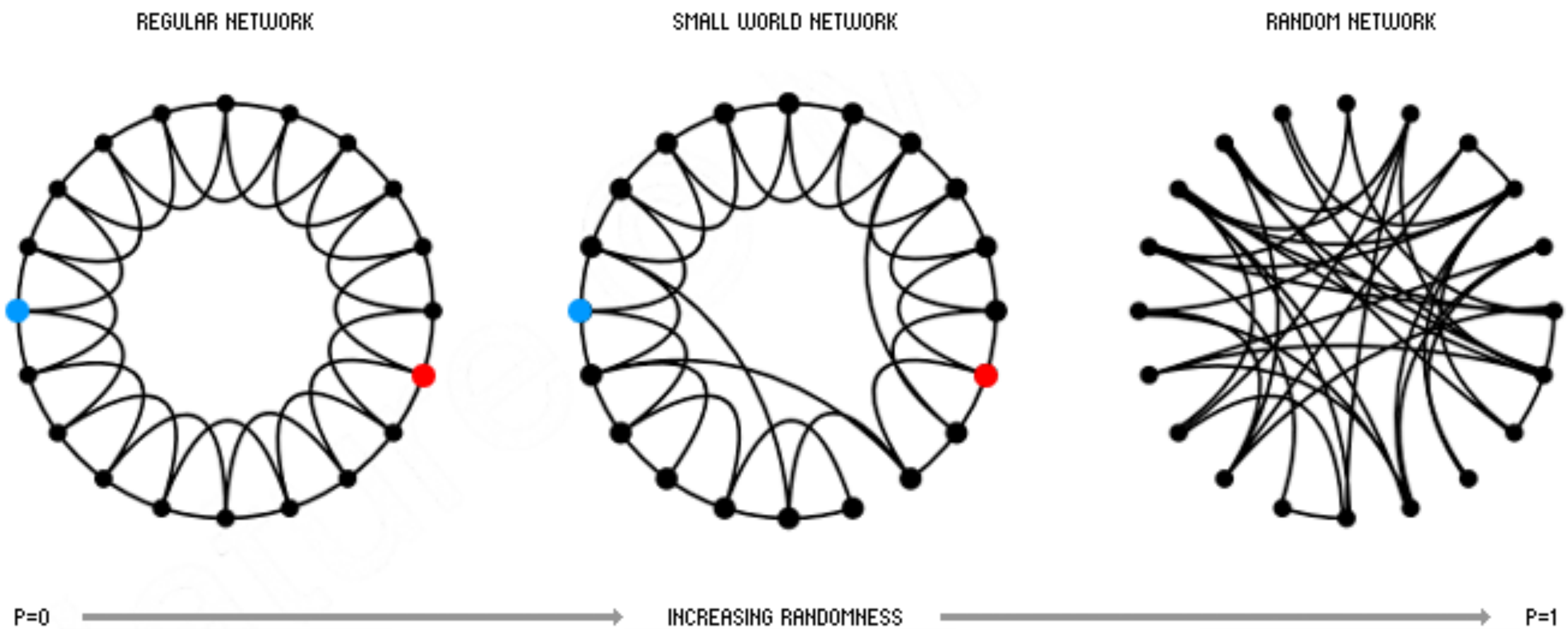
Two components to the model:

- **(1) Start with a low-dimensional regular lattice**
  - (In our case we are using a ring as a lattice)
  - Has high clustering coefficient
- **Now introduce randomness ("shortcuts")**
- **(2) Rewire:**
  - Add/remove edges to create shortcuts to join remote parts of the lattice
  - For each edge with prob. $p$ move the other end to a random node



# The Small-World Model

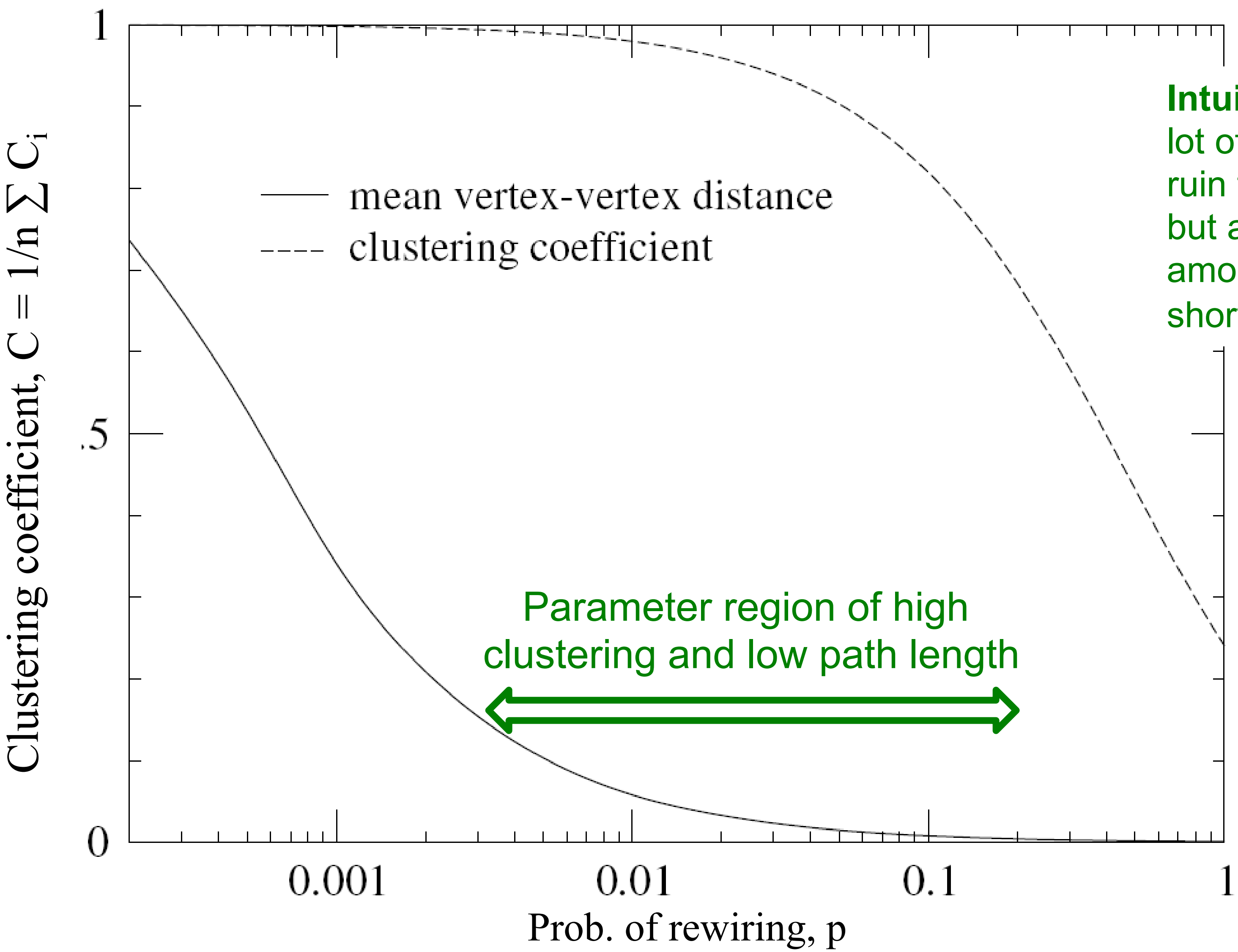
[Watts-Strogatz, '98]




| High clustering<br>High diameter | High clustering<br>Low diameter | Low clustering<br>Low diameter |
|---|---|---|
| $h = \frac{N}{2\bar{k}}$ $C = \frac{3}{4}$ | | $h = \frac{\log N}{\log \alpha}$ $C = \frac{\bar{k}}{N}$ |

Rewiring allows us to "interpolate" between a regular lattice and a random graph

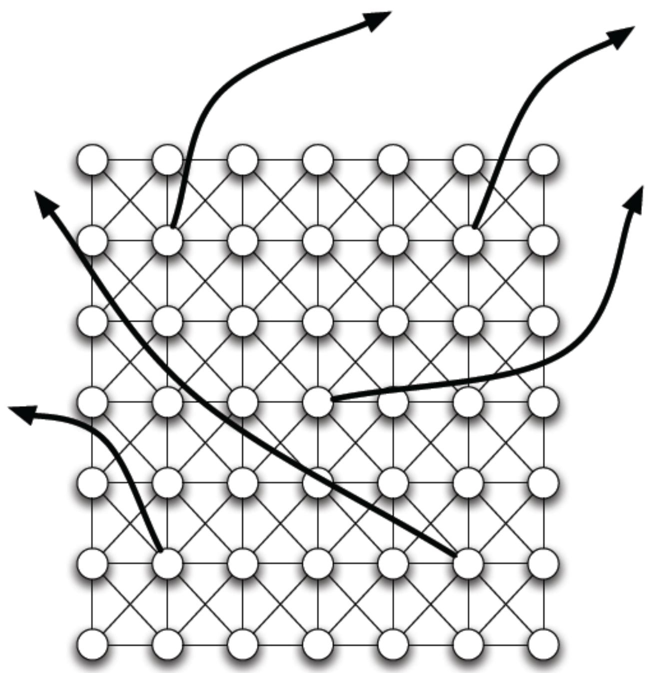
# The Small-World Model




**Intuition:** It takes a lot of randomness to ruin the clustering, but a very small amount to create shortcuts.

# Diameter of the Watts-Strogatz

- **Alternative formulation of the model:**
  - Start with a square grid
  - Each node has 1 random long-range edge
    - Each node has 1 spoke. Then randomly connect them.



$$C_i = \frac{2 \cdot e_i}{k_i(k_i - 1)} = \frac{2 \cdot 12}{9 \cdot 8} \geq 0.33$$

There are already 12 triangles in the grid and the long-range edge can only close more.

**What's the diameter?**

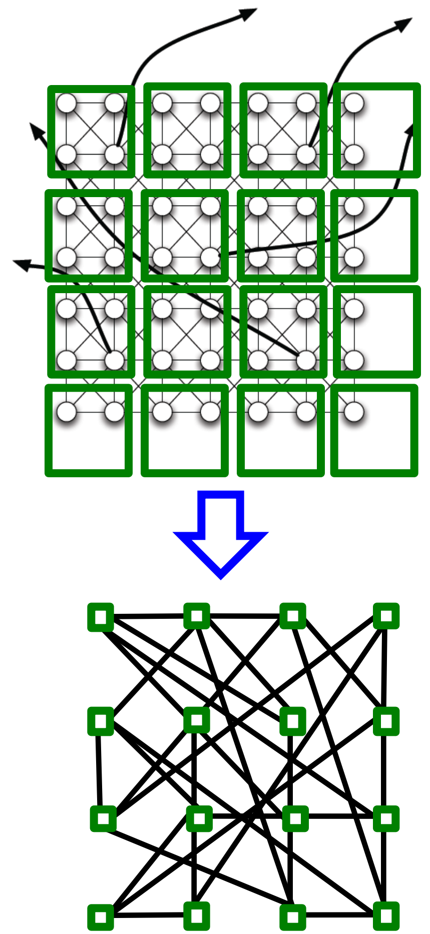
**It is *O(log(n))***

**Why?**

# Diameter of the Watts-Strogatz

- **Proof:**
  - Consider a graph where we contract 2x2 subgraphs into supernodes
  - Now we have 4 edges sticking out of each supernode
    - **4-regular random graph!**
  - From Thm. we have short paths between super nodes
  - We can turn this into a path in a real graph by adding at most 2 steps per long range edge (by having to traverse internal nodes)

⇒ **Diameter of the model is** ***O(2 log n)***



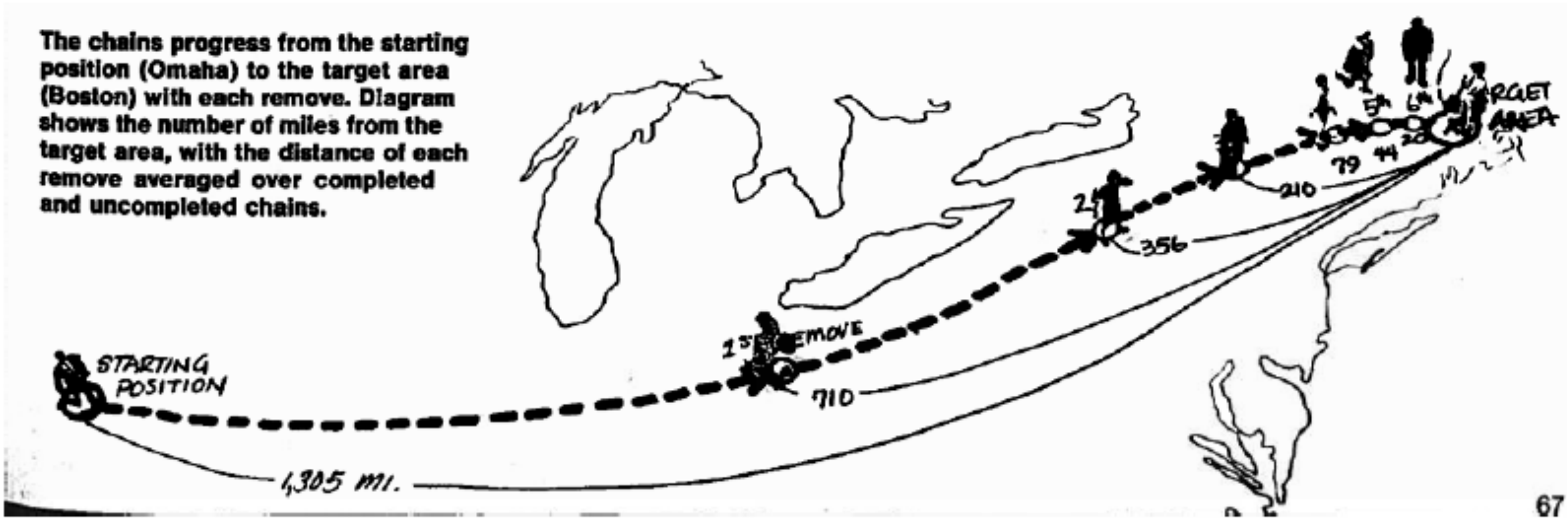
**4-regular random graph**

# Small-World: Summary

- **Could a network with high clustering be at the same time a small world?**
  - Yes! You don't need more than a few random links
- **The Watts Strogatz Model:**
  - Provides insight on the interplay between clustering and the small-world
  - Captures the structure of many realistic networks
  - Accounts for the high clustering of real networks
  - Does not lead to the correct degree distribution
  - Does not enable **navigation** **(next)**

# How to Navigate a Network?

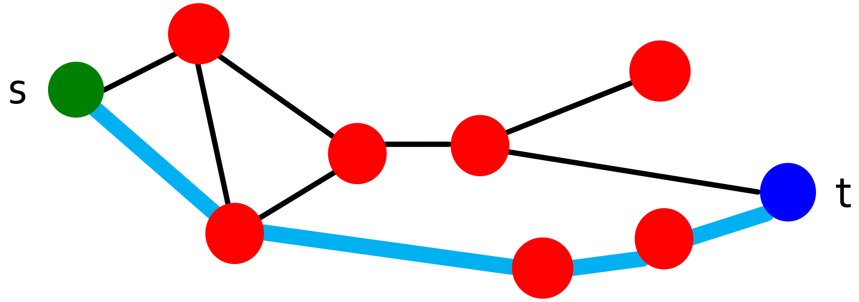
- **What mechanisms do people use to navigate networks and find the target?**



# Decentralized Search

**The setting:**

- $s$ only knows locations of its friends and location of the target $t$
- $s$ does not know links of anyone else but itself
- **Geographic Navigation:** $s$ "navigates" to a node geographically closest to $t$
- **Search time T:** Number of steps to reach $t$



# Overview of the Results

## Searchable

Search time T:

$$O((\log n)^{\beta})$$

**Kleinberg's model**

$$O((\log n)^{2})$$

## Not searchable

Search time T:

$$O(n^{\alpha})$$

**Watts-Strogatz**

$$O(n^{\frac{2}{3}})$$

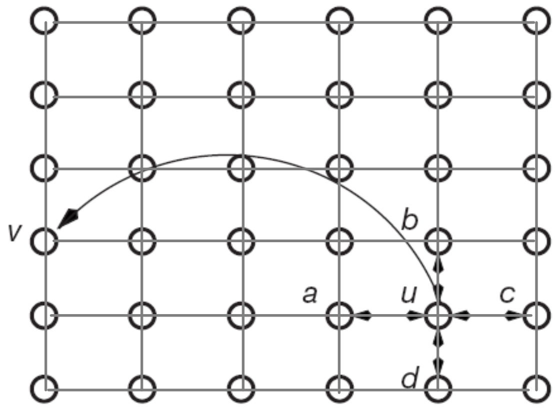
**Erdős–Rényi**

$$O(n)$$

**Note:** We know these graphs have diameter *O(log n)*.
So in Kleinberg's model search time is polynomial in *log n*,
while in Watts-Strogatz it is exponential (in *log n*).

# Navigation in Watts-Strogatz

- **Model:** 2-dim grid where each node has 1 random edge
  - This is a small-world!
    - **(Small-world = diameter O(log n))**

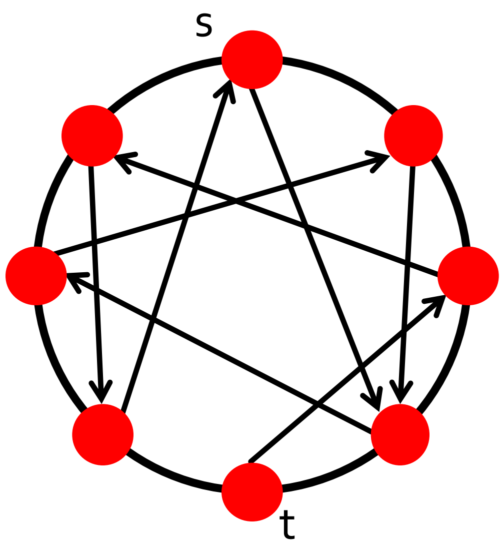



- **Fact:** A decentralized search algorithm in Watts-Strogatz model needs $n^{2/3}$ steps to reach $t$ in expectation
  - **Note:** Even though paths of $O(\log n)$ steps exist

- **Note:** All our calculations are asymptotic, i.e., we are interested in what happens as $n \to \infty$

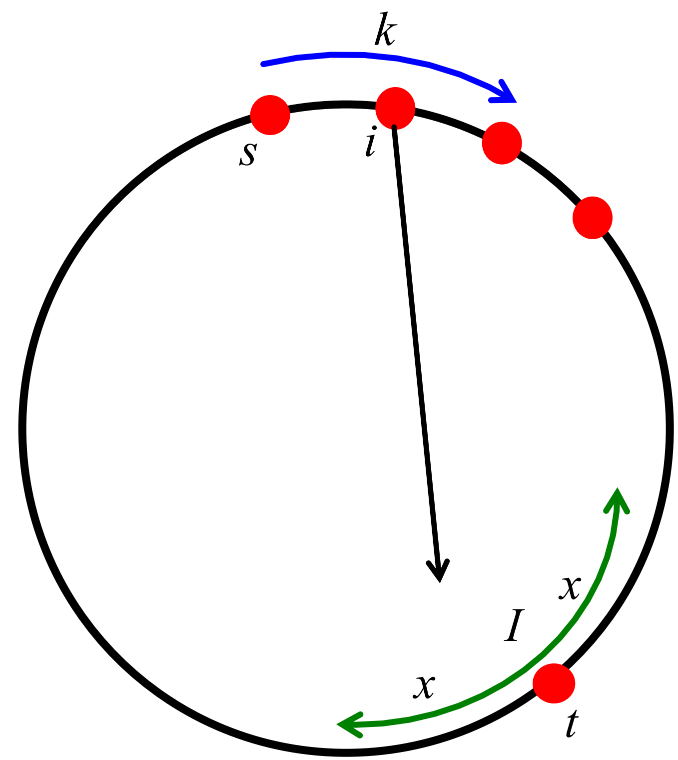
# Navigation in Watts-Strogatz

- **Let's do the proof for 1-dimensional case**
- **Want to show Watts-Strogatz is NOT searchable**
  - Bound the search time from below
- **About the proof:**
  - **Setting:** $n$ nodes on a ring plus one random directed edge per node.
  - Search time is $T \geq O(\sqrt{n})$
    - For **d**-dim. case: $T \geq O(n^{d/(d+1)})$
  - **Proof strategy: Principle of deferred decision**
    - Doesn't matter when a random decision is made if you haven't seen it yet
    - Assume random long range links are only created once you get to them



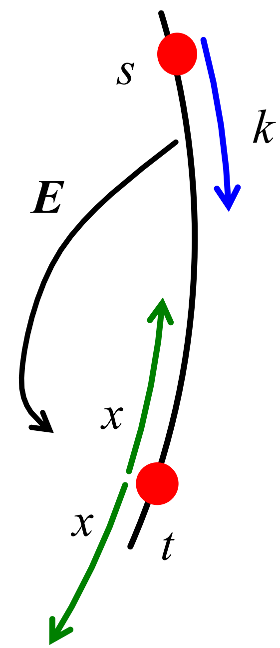
# Proof: Search time is ≥ O($n^{1/2}$)

- **Claim:**
  - **Expected search time is** $\geq \frac{1}{4}\sqrt{n}$
- **Let:** $E_i$ = event that long link out of node $i$ points to some node in interval $I$ of width $2{\cdot}x$ nodes (for some $x$) around target $t$
- **Then:** $P(E_i) = \frac{2x}{n-1} \approx \frac{2x}{n}$ (in the limit of large $n$)
  (haven't seen node $i$ yet, but can assume random edge generation)



# Proof: Search time is $\geq O(n^{1/2})$

- $\boldsymbol{E}$ = event that any of the first $\boldsymbol{k}$ nodes search algorithm visits has a link to $\boldsymbol{I}$
- **Then:** $P(E) = P\left(\bigcup_i^k E_i\right) \leq \sum_i^k P(E_i) = k\frac{2x}{n}$
- **Let's choose** $k = x = \frac{1}{2}\sqrt{n}$




**Then:**

$$P(E) \leq 2\frac{\left(\frac{1}{2}\sqrt{n}\right)^2}{n} = \frac{1}{2}$$

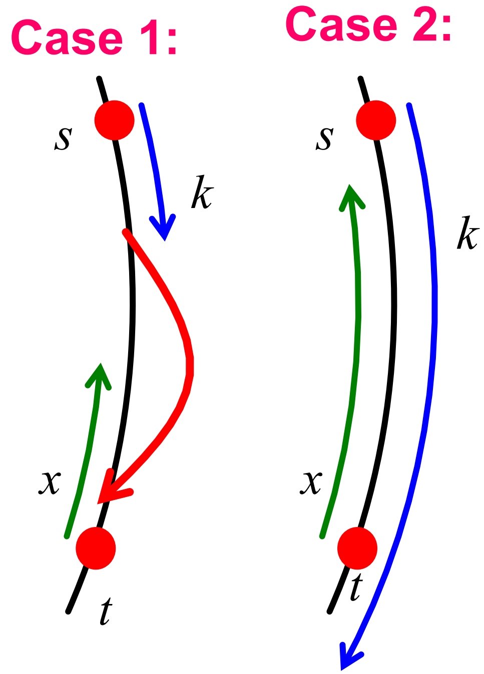
**Note:** Our alg. is deterministic and will choose to travel via a long- or short-range links using some deterministic rule.

The principle of deferred decision tells us that it does not really matter how we reached node $i$.
Its prob. of linking to interval $I$ is: $2x/n$.

# Proof: Search time is $\geq O(n^{1/2})$

**P(E)** = P(in $\frac{1}{2}\sqrt{n}$ steps we jump inside $\frac{1}{2}\sqrt{n}$ of $t$) $\leq \frac{1}{2}$

- **Suppose** initial $\boldsymbol{s}$ is outside $\boldsymbol{I}$ and event $\boldsymbol{E}$ does not happen (i.e., first $k$ visited nodes don't point to $\boldsymbol{I}$)
- **Then** the search algorithm must take $\boldsymbol{T \geq min(k, x)}$ steps to get to $\boldsymbol{t}$
  - **(1)** Right after we visit $\boldsymbol{k}$ nodes a good long-range link may occur
  - **(2)** $\boldsymbol{x}$ and $\boldsymbol{k}$ "overlap", due to $\boldsymbol{E}$ not happening we have to walk at least $\boldsymbol{x}$ steps

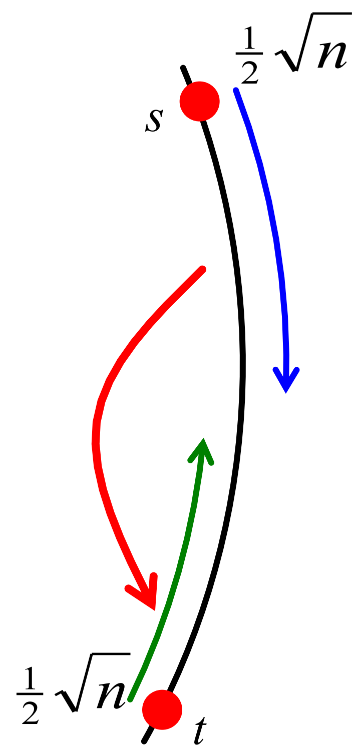


# Proof: Search time is $\geq O(n^{1/2})$

- **Claim:** Getting from $s$ to $t$ takes $\geq \frac{1}{4}\sqrt{n}$ steps
- *Search time ≥ P(E)(#steps) + P(not E) min(x,k)*
- **Proof:** We just need to put together the facts
  - **We already showed that for $x = k = \frac{1}{2}\sqrt{n}$**
    - $E$ does not happen with prob. ½
    - If $E$ does not happen,
      we must traverse $\geq \frac{1}{2}\sqrt{n}$ steps to get to $t$
  - **The expected time to get to $t$ is then**

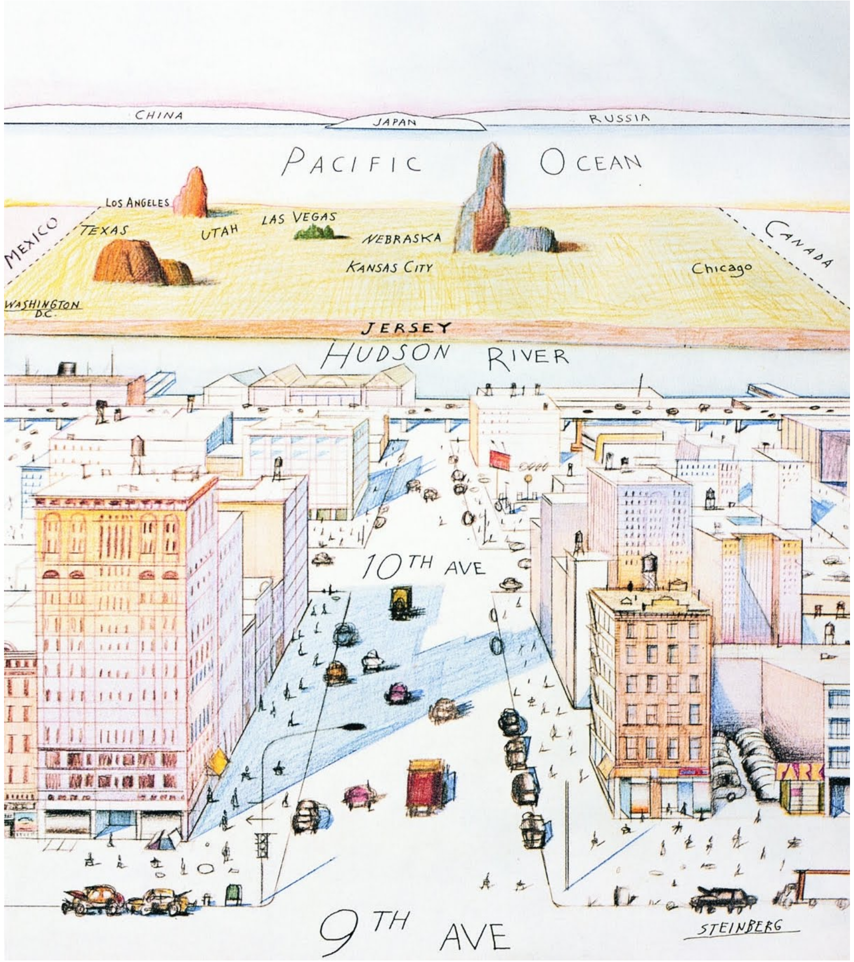
$$\geq P(E\ doesn't\ occur) \cdot \min\{x,k\} =$$

$$= \frac{1}{2}\frac{1}{2}\sqrt{n} = \frac{1}{4}\sqrt{n}$$



# Navigable Small-World Graph?

- Watts-Strogatz graphs are **not searchable**
- **How do we make a searchable small-world graph?**
- **Intuition:**
  - Our long range links are not random
  - **They follow geography!**



Saul Steinberg, “View of the World from 9th Avenue”