Power Laws in Networks

CS 322: (Social and Information) Network Analysis
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Announcements

- Return the homework after the class
- Project proposals are due in 1 week
- 2 parts to the proposal:
  - Reaction paper:
    - Read related papers, comment on them
    - Weaknesses, extensions, etc.
  - Proposed work:
    - Put your proposed work in the context of the papers you read
Degree distribution on the Web

In-degree (May 99, Oct 99) distr.

Out-degree (May 99, Oct 99) distr.
[Faloutsos, Faloutsos and Faloutsos, 1999]
[Barabasi-Albert, 1999]

- Actor collaborations
- Web graph
- Power-grid
- Take real network plot a histogram of $p_k$ vs. $k$

\[
Y = x^{-\alpha} \quad \text{or} \quad Y = e^{-x} \\
\log Y = -\lambda \log X
\]

Flickr social network
$n = 584,207, \quad m = 3,555,115$
Plot the same data on log-log axis:

Flickr social network

\[ n = 584,207, \quad m = 3,555,115 \]
Exponential tail vs. Power-law tail

Power law: \( Y \sim X^{-2} \)
Exponential: \( Y \sim e^{-X} \)
Many other quantities follow heavy-tailed distributions
Not everyone likes power-laws 😊

CMU students protesting at the G20 meeting in Pittsburgh in Sept 2009
Power law degree exponent is typically $2 < \alpha < 3$

- Web graph [Broder et al. 00]:
  - $\alpha_{in} = 2.1$, $\alpha_{out} = 2.4$
- Autonomous systems [Faloutsos et al. 99]:
  - $\alpha = 2.4$
- Actor collaborations [Barabasi-Albert 00]:
  - $\alpha = 2.3$
- Citations to papers [Redner 98]:
  - $\alpha \approx 3$
- Online social networks [Leskovec et al. 07]:
  - $\alpha \approx 2$
What is the normalizing constant?

\[ p(x) = c \cdot x^{-\alpha} \quad C = ? \]

\[ \int_{x_{\text{min}}}^{\infty} p(x) \, dx = C \int_{x_{\text{min}}}^{\infty} x^{-\alpha} \, dx \]

\[ = \frac{C}{\alpha-1} \left[ x^{\alpha-1} \right]_{x_{\text{min}}}^{\infty} \]

\[ C = \left( \frac{\alpha-1}{\alpha-1} \right) x_{\text{min}}^{\alpha-1} \]
### Mathematics of Power-laws

- \( E[x] = \int_{x_{\min}}^{\infty} x \, p(x) \, dx = \int_{x_{\min}}^{\infty} x^{-\alpha + 1} \, dx \)

  \[
  = \frac{C}{2-\alpha} \left[ x^{2-\alpha} \right]_{x_{\min}}^{\infty}
  \]

- **Tails are heavy:**
  - If \( \alpha \leq 2 \) : \( E[x] = \infty \)
  - If \( \alpha \leq 3 \) : \( \text{Var}[x] = \infty \)

  \( x_i \) is degree of node \( i \)
Estimating $\alpha$ from data:

1. Fit a line on log-log axis using least squares

![Log-log plot with fitted line]

[Clauset-Shalizi-Newman 2007]
Estimating $\alpha$

- Estimating $\alpha$ from data:
  
  2. Plot Complementary CDF $P(X>x)$

  Then $\alpha = 1 + \alpha^*$ where $\alpha^*$ is the slope of $P(X>x)$.

  e.i., if $P(X=x) \propto x^{-\alpha}$ then $P(X>x) = x^{-(\alpha-1)}$

  $P(X>x) = \sum_{j=x}^{\infty} p(j) = \int_{x}^{\infty} c_j j^{-\alpha} \, dj$

  $= \frac{c}{\alpha} x^{-(\alpha-1)}$
Estimating power-law exponent $\alpha$ from data:

3. Use MLE:

$$1 + n \left[ \sum_{i=1}^{n} \ln \frac{x_i}{x_{\text{min}}} \right]^{-1}$$

$x_i$ is degree of node $i$

$$\eta(x) = \frac{x^{-\alpha}}{x_{\text{min}}^{-\alpha}} \left( \frac{x}{x_{\text{min}}} \right)^{-\alpha}$$

$$L(\alpha) = \log \prod_i n(x_i) = \sum_i \log n(x_i)$$

$$= \frac{m}{2} \left[ \log (\alpha - 1) - \log x_{\text{min}} + \log \left( \frac{x_i}{x_{\text{min}}} \right) \right]$$

$$= \frac{m}{2} \left[ \log (\alpha - 1) - \log x_{\text{min}} + \log \left( \frac{x_i}{x_{\text{min}}} \right) \right]$$

$$\frac{\alpha x}{x_{\text{min}}} = 0 \Rightarrow \frac{m}{(\alpha - 1)} - \sum_i \log \left( \frac{x_i}{x_{\text{min}}} \right) = \alpha = 1 + m \left[ \log \left( \frac{x_i}{x_{\text{min}}} \right) \right]$$
Flickr: Degree exponent

- Linear scale

- Log scale, $\alpha=1.75$

- CCDF, Log scale, $\alpha=1.75$

- CCDF, Log scale, $\alpha=1.75$, exp. cutoff
Can not arise from sums of independent events

- Recall: in $G_{np}$ each pair of nodes in connected independently with prob. $p$

$$x_i = \deg v_i \text{ node } i$$

$$\mathbb{E}(x_i) = \mathbb{E}(\sum_{i=1}^{m} x_i) = \sum_{i=1}^{m} \mathbb{E}[x_i] = \mu (m-1)$$
Random network
(Erdos-Renyi random graph)
Degree distribution is Binomial

Scale-free (power-law) network
Degree distribution is Power-law
Function is scale free if:
\[ f(ax) = c f(x) \]
What is a good model that gives rise to power-law degree distributions?

What is the analog of central limit theorem for power-laws?
Model: Preferential attachment

- Preferential attachment
  [Price 1965, Albert-Barabasi 1999]:
  - Nodes arrive in order
  - A new node creates $m$ out-links
  - Prob. of linking to a previous node $i$ is proportional to its degree $d_i$

$$P(j \rightarrow i) = \frac{d_i}{\sum_k d_k}$$
Rich-get-richer

- New nodes are more likely to link to nodes that already have high degree

- Herbert Simon’s result
  - Power-laws arise from “Rich get richer” (cumulative advantage)

- Examples [Price 65]:
  - Citations: new citations of a paper are proportional to the number it already has
Pages are created in order 1,2,3,…,N

When page \( j \) is created it produces a link to earlier webpages:

1) With prob. \( p \), page \( j \) creates a link to a page \( i \) chosen uniformly at random (from among all earlier pages)

2) With prob. \( 1-p \), page \( j \) chooses a page \( i \) uniformly at random (from among all earlier pages) and creates a link to \( \text{the page } i \text{ points to.} \)

Note this is same as saying:

2) With prob. \( 1-p \), page \( j \) creates a link to the page \( i \) with prob. proportional to \( d_i \) (the degree of \( i \))
Claim: the described model generates networks where the fraction of nodes with degree \( d \) scales as:

\[
\ln \left(1 + \frac{1}{d^2}\right)
\]

where

\[
q = 1 - \mu
\]
Continuous approximation (1)

- Degree $d_i(t)$ of node $i$ ($i=1,2,...,n$) is a continuous quantity and it grows deterministically as a function of time $t$.

- Analyze $d_i(t)$ – continuous degree of node $i$ at time $t \geq i$. 
What do we know?

- Initial condition: \( d_i(t) = 0 \), when \( t = i \) (\( i \) just arrived)
- Expected change of \( d_i(t) \) over time:
  - Node \( i \) gains an in-link at step \( t+1 \) only if a link from a newly created node \( t+1 \) points to it.
  - What’s the prob. of this event?
    - With prob. \( p \) node \( t+1 \) links to a random node: links to \( i \) with prob. \( 1/t \)
    - With prob. \( 1-p \) node \( t+1 \) links preferentially: links to \( i \) with prob. \( d_i(t)/t \)
    - So: prob. node \( t+1 \) links to \( i \) is

\[ \gamma \frac{1}{t} + (1-\gamma) \frac{d_i(t)}{2t} \]
At time $t$ we have $t$ nodes

What is the rate of growth of $d_i$?

$$\frac{d d_i}{d t} = \mu \frac{1}{t} + (1-\rho) \frac{d_i}{t}$$

$$\frac{d d_i}{d t} = \rho + q \frac{d_i}{t}$$

$q = 1 - \rho$
What is the rate of growth of \( d_i \)?

\[
\frac{d d_i}{d t} = \frac{n + q d_i}{t} = \frac{1}{n + q d_i} \frac{d d_i}{d t} = \frac{1}{t}
\]

\[
\int \frac{1}{n + q d_i} \frac{d d_i}{d t} \, d t = \int \frac{1}{t} \, d t \Rightarrow \log (n + q d_i) = q \log t + c
\]

\[n + q d_i = At^2 \quad A = e^c\]

\[d_i(t) = \frac{1}{2} (\Delta t^2 - n)\]
What is the constant A?

- We know: \( d_i(t) = 0 \)

\[
\frac{1}{A} (\Lambda i^k - \mu) = 0 \quad \Rightarrow \quad A = \frac{\mu}{i^k}
\]

\[
d_i(t) = \frac{1}{A} \left[ \frac{\mu}{i^k} t^k - \mu \right]
\]

\[
d_i(t) = \frac{\mu}{A} \left[ (\frac{t}{i})^k - 1 \right]
\]
Degree distribution

- Fraction of nodes with degree \( > d \) at time \( t \)

\[
\frac{\sqrt{n}}{\lambda} \left[ \left( \frac{t}{t} \right)^{\frac{1}{2}} - 1 \right] > d
\]

\[
\Rightarrow \quad t \geq \frac{\frac{2}{\lambda} \cdot d + 1}{\frac{2}{\lambda} - 1}
\]

\#nodes fraction
Degree distribution

- Fraction of nodes with degree exactly \( d \) at time \( t \)?

\[
\left[ \frac{x}{d+1} \right]^{-\frac{d}{2}} \frac{d}{x} \sqrt{\frac{x}{d+1} - 1} \left( 1 + \frac{1}{d} \right)
\]

\[
= \frac{1}{2} \sum \left[ \frac{x}{d+1} \right]^{-\frac{d}{2}} \sqrt{\frac{x}{d+1} - 1} \left( 1 + \frac{1}{d} \right)
\]

\[
\Rightarrow \lambda = 1 + \frac{1}{d} = 1 + \frac{1}{(d+1)}
\]