CS246: Probability Review Session

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1. Random variables, Probability distributions

2. Working with probabilities

3. Bayes

4. Expected Value and Variance

5. Useful things for HW1

Random variables

Definition

- A set of values $X_1, \ldots X_n$
- Probabilities $P(X_1), \ldots P(X_n)$ (all $\leqslant 1$, $\sum_i P(X_i) = 1$)

Examples

- Dice: $X = \{1, 2, , \ldots 6\}$, uniform probabilities.
- Outcome of next election: $X = \{\text{Democrat}, \text{Republican}\}$, probabilities ?

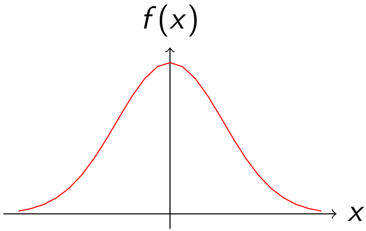
Events: $X = 1$, X is odd, ...

Continuous random variables

- Infinite set of values, e.g. $[0, 1]$, $\mathbb{R}$.
- Probability Distribution Function (pdf) $f : X \to \mathbb{R}_+$

$$\int_X f(x)dx = 1$$

Example: Gaussian distribution



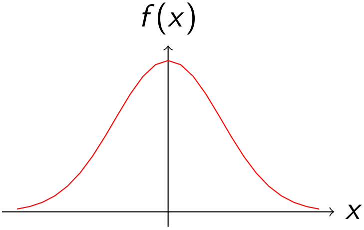
Continuous random variables

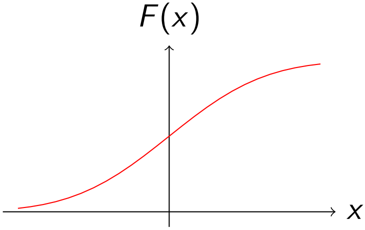
Cumulative Distribution Function (cdf)

$$F(x) = \int_{-\infty}^{x} f(u)du$$

- $0 \leqslant F(x) \leqslant 1$
- $F \nearrow$

Example: Gaussian distribution





Common distributions

Examples

- Normal: $X = \mathbb{R}$

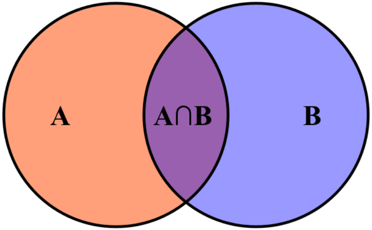
$$f(x) = \frac{1}{\sqrt{2\sigma\pi}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

- Bernoulli: $X = \{0, 1\}$. $P(1) = p, P(0) = 1 - p$
- Uniform: $X = [a, b]$. $\forall x \in \mathbb{R}, f(x) = \frac{1}{b-a}\mathbf{1}(a \leqslant x \leqslant b)$
- Geometric: $X = \{0, 1, \dots\}$

$$P(n) = p(1-p)^{n-1}$$

The basic rules

- $P(\Omega) = 1$
- $\forall A \subset \Omega, P(A) \geqslant 0$
- $\forall A, B \subset \Omega, P(A \cup B) = P(A) + P(B) - P(A \cap B)$



- $\forall A \subset \Omega, P(\neg A) = 1 - P(A)$
- $\forall A, B \subset \Omega, A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$

Conditional Probabilities

Definition:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Independent events

$$\begin{aligned} A \perp B &\Leftrightarrow P(A \cap B) = P(A)P(B) \\ &\Leftrightarrow P(A|B) = P(A) \end{aligned}$$

Example

For a couple with two childrens, the probability $P(\text{two daughters}|\text{at least one daughter})$ is:

- < 0.5
- $= 0.5$
- > 0.5

Answer: < 0.5!

Example

Events: G_1 = first child is a girl, G_2 = second child is a girl.

$$
\begin{aligned}
&P(\text{two daughters} | \geqslant \text{one daughter}) \\
&= P(G_1 \cap G_2 | G_1 \cup G_2) \\
&= \frac{P(G_1 \cap G_2 \cap (G_1 \cup G_2))}{P(G_1 \cup G_2)} \\
&= \frac{P(G_1 \cap G_2)}{P(G_1) + P(G_2) - P(G_1 \cap G_2)} \\
&= \frac{0.25}{0.5 + 0.5 - 0.25} \\
&= \frac{1}{3}
\end{aligned}
$$

Bayes

Inversing causality

Goal: Knowing $P(\text{symptom}|\text{disease})$, compute $P(\text{disease}|\text{symptom})$.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\neg A)P(\neg A)}$$

Variables: H = has HIV. T = positive test.

Given:

- $P(H) = 0.16$ %
- $P(T|H) = 99.7$ %
- $P(\neg T|\neg H) = 98.5$ %

$$
\begin{aligned}
P(H|\neg T) &= \frac{P(\neg T|H)P(H)}{P(\neg T)} \\
&= \frac{P(\neg T|H)P(H)}{P(\neg T|H)P(H) + P(\neg T|\neg H)P(\neg H)} \\
&= \frac{0.3 \times 0.16}{0.3 \times 0.16 + 98.5 \times 99.84} \\
&= 4.9\mathrm{E}^{-6}
\end{aligned}
$$

Bayes

Example

$$
\begin{aligned}
P(\neg H|T) &= \frac{P(T|\neg H)P(\neg H)}{P(T)} \\
&= \frac{P(T|\neg H)P(\neg H)}{P(T|\neg H)P(\neg H) + P(T|H)P(H)} \\
&= \frac{1.5 \times 99.84}{1.5 \times 99.84 + 99.7 \times 0.16} \\
&= 90.4\%
\end{aligned}
$$

Expected Value

Definition

$$\mathbb{E}(X) = \sum_i X_i P(X_i), \text{ or } \int_x x f(x) dx$$

Properties

- $\forall a, b \in \mathbb{R}, E(aX + b) = a\mathbb{E}(X) + b$
- $\mathbb{E}(X + Y) = E(X) + E(Y)$

Variance

Definition

$$\mathbb{V}(X) = \mathbb{E}\left[(X - \mathbb{E}(X))^2\right]$$

An easier way to compute it

$$\begin{aligned}
\mathbb{V}(X) &= \mathbb{E}\left[(X - \mathbb{E}(X))^2\right] \\
&= \mathbb{E}\left[X^2 - 2X\mathbb{E}(X) + \mathbb{E}(X)^2\right] \\
&= \mathbb{E}(X^2) - 2\mathbb{E}(X)^2 + \mathbb{E}(X)^2 \\
&= \mathbb{E}(X^2) - \mathbb{E}(X)^2
\end{aligned}$$

Markov Inequality

Needed for problem set 1

X r.v. in $\mathbb{R}$

$$\forall a > 0, P(|X| > a) \leqslant \frac{\mathbb{E}(|X|)}{a}$$

Proof:
Define r.v. $Y = a$ if $|X| > a$, 0 otherwise. $|X| \geqslant Y$.
Thus, $\mathbb{E}(|X|) \geqslant \mathbb{E}(Y)$.

$$\begin{aligned}\mathbb{E}(Y) &= aP(Y = a) + 0 \times P(Y = 0) \\ &= aP(Y = a) \\ &= aP(|X| > a)\end{aligned}$$