Quick Tour of Basic Linear Algebra

CS246: Mining Massive Data Sets
Winter 2012

Matrices and Vectors

- Matrix: A rectangular array of numbers, e.g., $A \in \mathbb{R}^{m \times n}$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

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- Vector: A matrix consisting of only one column (default) or one row, e.g., $x \in \mathbb{R}^n$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Matrix Multiplication

- If $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $C = AB$, then $C \in \mathbb{R}^{m\times p}$:

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- Special cases: Matrix-vector product, inner product of two vectors. e.g., with $x, y \in \mathbb{R}^n$:

$$x^T y = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}$$

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- Block multiplication: If $A = [A_{ik}]$, $B = [B_{kj}]$, where A_{ik}'s and B_{kj}'s are matrix blocks, and the number of columns in A_{ik} is equal to the number of rows in B_{kj}, then $C = AB = [C_{ij}]$ where $C_{ij} = \sum_k A_{ik} B_{kj}$

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  Example: If $\overrightarrow{x} \in \mathbb{R}^n$ and $A = [\overrightarrow{a_1} | \overrightarrow{a_2} | \ldots | \overrightarrow{a_n}] \in \mathbb{R}^{m \times n}$, $B = [\overrightarrow{b_1} | \overrightarrow{b_2} | \ldots | \overrightarrow{b_p}] \in \mathbb{R}^{n \times p}$:

$$A\overrightarrow{x} = \sum_{i=1}^{n} x_i \overrightarrow{a_i}$$

$$AB = [A\overrightarrow{b_1} | A\overrightarrow{b_2} | \ldots | A\overrightarrow{b_p}]$$

Operators and properties

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  - $tr(A + B) = tr(A) + tr(B)$
  - $tr(\lambda A) = \lambda tr(A)$
  - If AB is a square matrix, $tr(AB) = tr(BA)$

Special types of matrices

- Identity matrix: $I = I_n \in \mathbb{R}^{n \times n}$:

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- Orthogonal matrices: $U \in \mathbb{R}^{n\times n}$ is orthogonal if $UU^T = I = U^T U$

Linear Independence and Rank

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- Properties:
  - $rank(A) \leq \min\{m, n\}$
  - $rank(A) = rank(A^T)$
  - $rank(AB) \leq \min\{rank(A), rank(B)\}$
  - $rank(A + B) \leq rank(A) + rank(B)$

Matrix Inversion

- If $A \in \mathbb{R}^{n \times n}$, $rank(A) = n$, then the inverse of A, denoted A^{-1} is the matrix that: $AA^{-1} = A^{-1}A = I$
- Properties:
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(A^{-1})^T = (A^T)^{-1}$

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- Nullspace: $null(A) = \{x \in \mathbb{R}^n | \, Ax = 0\}$

Determinant

- $A \in \mathbb{R}^{n \times n}$, $a_1, \ldots, a_n$ the rows of A, $S = \{\sum_{i=1}^{n} \alpha_i a_i | \, 0 \leq \alpha_i \leq 1\}$, then $det(A)$ is the volume of S.
- Properties:
  - $det(I) = 1$
  - $det(\lambda A) = \lambda^n det(A)$
  - $det(A^T) = det(A)$
  - $det(AB) = det(A)det(B)$
  - $det(A) \neq 0$ if and only if A is invertible.
  - If A invertible, then $det(A^{-1}) = det(A)^{-1}$

Quadratic Forms and Positive Semidefinite Matrices

- $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$, $x^T A x$ is called a quadratic form:

$$x^T A x = \sum_{1 \le i,j \le n} A_{ij} x_i x_j$$

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- A is positive definite if $\forall\, x \in \mathbb{R}^n : x^T A x > 0$
- A is positive semidefinite if $\forall\, x \in \mathbb{R}^n : x^T A x \geq 0$
- A is negative definite if $\forall\, x \in \mathbb{R}^n : x^T A x < 0$
- A is negative semidefinite if $\forall\, x \in \mathbb{R}^n : x^T A x \leq 0$

Eigenvalues and Eigenvectors

- $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{C}$ is an eigenvalue of A with the corresponding eigenvector $x \in \mathbb{C}^n$ ($x \neq 0$) if:

$$Ax = \lambda x$$

- eigenvalues: the n possibly complex roots of the polynomial equation $det(A - \lambda I) = 0$, and denoted as $\lambda_1, \ldots, \lambda_n$
- Properties:
  - $tr(A) = \sum_{i=1}^{n} \lambda_i$
  - $det(A) = \prod_{i=1}^{n} \lambda_i$
  - $rank(A) = |\{1 \leq i \leq n | \lambda_i \neq 0\}|$

Matrix Eigendecomposition

- $A \in \mathbb{R}^{n \times n}$, $\lambda_1, \ldots, \lambda_n$ the eigenvalues, and $x_1, \ldots, x_n$ the eigenvectors. $X = [x_1|x_2|\ldots|x_n]$, $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$, then $AX = X\Lambda$.
- A called diagonalizable if X invertible: $A = X\Lambda X^{-1}$
- If A symmetric, then all eigenvalues real, and X orthogonal (hence denoted by $U = [u_1|u_2|\ldots|u_n]$):

$$A = U\Lambda U^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

- A special case of Signular Value Decomposition

Example: Random Walks

- Graph $G = (V, E)$, with $V = \{1, 2, \ldots, n\}$, and $|E| = m$.
- Stationary distribution satisfies:

$$\pi_v = \sum_{(w,v) \in E} \frac{\pi_w}{\deg_w}$$

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- Let $A \in \mathbb{R}^{n \times n}$, with

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- For undirected graphs: $\pi_v = \frac{\deg_v}{2m}$

Example: PageRank

- At each step, jump to a random node with probability ϵ
- PageRank satisfies:

$$\pi_v = (1-\epsilon)\sum_{(w,v)\in E}\frac{\pi_w}{\deg_w} + \frac{\epsilon}{n}$$

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- In matrix form: $\pi = (1-\epsilon)A\pi + \frac{\epsilon}{n}\overrightarrow{1}$
- $\pi = \frac{\epsilon}{n}[I - (1-\epsilon)A]^{-1}\overrightarrow{1}$