The expansion of random regular graphs

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Introduction

Our aim is now to show that for any $d \geq 3$, *almost all* d-regular graphs on $\{1, 2, \ldots, n\}$ have edge-expansion ratio at least $c_d d$ (if nd is even), where $c_d > 0$ depends only upon d. (In fact, we will see that we can take $c_d \geq 0.18$ for all $d \geq 3$, and $c_d \to 1/2$ as $d \to \infty$.) More precisely, if $G(n, d)$ denotes a *(uniform) random d-regular graph on* $[n]$, meaning a (labelled) d-regular graph on $[n]$ chosen uniformly at random from the set of all d-regular graphs on $[n]$, then

$$\mathbb{P}\{h(G_{n,d}) \geq c_d d\} \to 1 \quad \text{as } n \to \infty.$$

We will work only with *labelled* n-vertex graphs; from now on, 'a graph on $[n]$' will always mean a *labelled* graph on $[n]$.

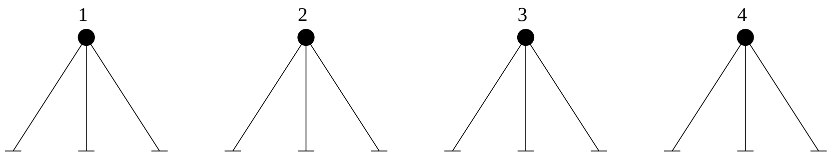
Recall that if $A_1, A_2, \ldots$ is a sequence of events in probability spaces $\Omega_1, \Omega_2, \ldots$, we say that '$A_n$ occurs *with high probability*' if $\mathbb{P}(A_n) \to 1$ as $n \to \infty$. In this language, we wish to prove that with high probability, $h(G(n, d)) \geq c_d d$, for any fixed $d \geq 3$. First, however, we need an 'efficient' way of sampling uniformly from the set of all d-regular graphs on $[n]$, and of analysing the properties of a uniform random d-regular graph on $[n]$.

It is easy to see that for any $d \in \mathbb{N}$ and any $n > d$ such that nd is even, there exist d-regular n-vertex graphs. The concept of a uniform random d-regular graph on $[n]$ is a simple one, but uniform random d-regular graphs are slightly harder to work with than Erdős-Renyi random graphs $G_{n,p}$. In $G_{n,p}$, each edge of $K_{[n]}$ is present *independently* with probability p, whereas in $G(n, d)$, no two edges are independent. (By which we mean, of course, the events $\{e \text{ is present}\}, \{f \text{ is present}\}$ are never independent.)

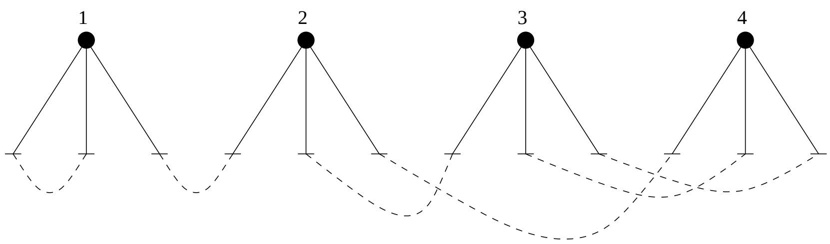
One way of generating a uniform random d-regular graph on $[n]$ is to take an Erdős-Renyi random graph $G_{n,d/(n-1)}$, and then condition on the event that it is d-regular. However, the probability that $G_{n,d/(n-1)}$ is d-regular tends to zero as $n \to \infty$, so this is not a useful generation method, and cannot be used to prove statements about almost all d-regular graphs. (Events that occur almost surely in $G_{n,d/(n-1)}$ will not necessarily occur almost surely in $G(n, d)$.) What we want is a random process that generates a d-regular graph on $[n]$ with probability bounded above by some positive constant depending only upon d. This is provided by Bollobás' *configuration model.*

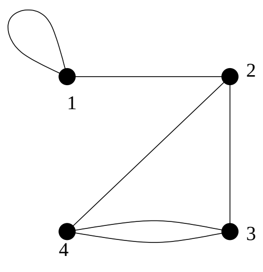
The configuration model

Let $d \in \mathbb{N}$, and let $n > d$ such that nd is even. Bollobás' *configuration model* produces a uniform random d-regular graph on $[n]$ as follows. Start with n vertices, labelled $1, 2, \ldots, n$, and draw d lines (called 'half-edges') emanating from each, so that the ends of the half-edges are all distinct. This produces n stars, or 'bunches', each consisting of d half-edges. For example, when $d = 3$ and $n = 4$, we have:



Let W be the set of 'ends' of the half-edges; index these ends by $[n] \times [d]$, where $(i, 1), (i, 2), \ldots, (i, d)$ are the ends emanating from the vertex i. Now choose a random matching $\mathcal{M}$ of the set W of 'ends' (uniformly at random from the set of all matchings of W), and use the edges of the random matching to join pairs of half-edges to produce edges. This produces a d-regular *multigraph* on $[n]$: it may have loops, if the matching $\mathcal{M}$ joins two ends that emanate from the same vertex, or multiple edges, if the matching has two or more edges between the same two 'bunches'. (Note that a loop contributes 2 to the degree of a vertex.) Here is a matching (consisting of the dotted edges), and the corresponding labelled 3-regular multigraph on $\{1, 2, 3, 4\}$:





Let $G^*(n, d)$ denote the (random) d-regular labelled multigraph on $[n]$ produced by this process. It is not a *uniform* random multigraph on $[n]$: the probability that a particular multigraph arises depends on the number of loops and on the mulitiplicities of the edges. However, it turns out that

$$\mathbb{P}\{G^*(n, d) \text{ is simple}\} \to e^{-(d^2-1)/4} \quad \text{as } n \to \infty.$$

So for any fixed $d \in \mathbb{N}$, with probability bounded away from 0 independently of n, this process yields a simple, labelled d-regular graph on $[n]$. Moreover, each *simple*[1] labelled d-regular graph on $[n]$ has the same probability of arising: indeed, it is easy to see that a simple d-regular graph on $[n]$ arises from precisely $(d!)^n$ of the matchings. Hence, if we condition on $G^*(n,d)$ being simple, the resulting distribution is the uniform distribution on the set of all d-regular graphs on $[n]$, i.e. we produce $G(n,d)$!

Observe that if $G^*(n,d)$ has a certain property $\mathcal{A}$ with high probability, then so does $G(n,d)$. Indeed, if $\mathbb{P}\{G^*(n,d) \notin \mathcal{A}\} \to 0$ as $n \to \infty$, then

$$\mathbb{P}\{G(n,d) \notin \mathcal{A}\} = \frac{\mathbb{P}\{G^*(n,d) \notin \mathcal{A},\ G^*(n,d) \text{ is simple}\}}{\mathbb{P}\{G^*(n,d) \text{ is simple}\}} \to 0 \qquad \text{as } n \to \infty.$$

The above argument yields the simplest known proof of an asymptotic formula for the number of labelled d-regular graphs on $[n]$. Note that if $m \in \mathbb{N}$ is even, the number $M(m)$ of matchings of an m-element set is

$$M(m) = (m-1)(m-3)\dots(3)(1) := (m-1)!!.$$

Clearly,

$$M(m) = \frac{m!}{2^{m/2}(m/2)!}.$$

Recall Stirling's Formula,

$$(m-1)! = (1+o(1))\sqrt{2\pi m}\ m^m e^{-m}.$$

It follows that

$$M(m) = (1+o(1))\sqrt{2}\ m^{m/2}e^{-m/2}. \tag{1}$$

Therefore, the number $N_{n,d}$ of labelled d-regular graphs on $[n]$ satisfies

$$\begin{aligned} N_{n,d} &= (1+o(1))e^{-(d^2-1)/4}\frac{M(nd)}{(d!)^n} \\ &= \sqrt{2}(1+o(1))e^{-(d^2-1)/4}\frac{(nd)^{nd/2}e^{-nd/2}}{(d!)^n} \\ &= (1+o(1))e^{-(d^2-1)/4}(d^{d/2}e^{-d/2}/d!)^n n^{nd/2}. \end{aligned}$$

We will now show that

$$\mathbb{P}\{G^*(n,d) \text{ is simple}\} \to e^{-(d^2-1)/4} \quad \text{as } n \to \infty.$$

Let the random variable $Z_{l,n}$ be the number of l-cycles in $G^*(n,d)$; so $Z_{1,n}$ is the number of loops, and $Z_{2,n}$ is the number of pairs of multiple edges. Formally, an l-cycle in $G^*(n,d)$ is a set of l distinct edges of the form

$$\{v_1v_2, v_2v_3, \dots, v_{l-1}v_l, v_lv_1\},$$

where $v_1, v_2, \dots, v_l \in [n]$ are distinct vertices.

If k is fixed and n is large, then although no two cycles of length at most k are independent (more precisely, their indicator functions are not independent), two

[1]Recall that a multigraph is said to be *simple* if it is a genuine graph, i.e. it has no loops or multiple edges.

vertex-disjoint cycles of length at most k are 'almost' independent, so most pairs of cycles of length at most k are 'almost' independent. In such circumstances, we can often use *Poisson approximation.*

Let $Y, X_1, X_2, \ldots$ be random variables, all taking integer values. We say that X_n *converges in distribution to* Y *as* $n \to \infty$ if for any $i \in \mathbb{Z}$, $\mathbb{P}\{X_n = i\} \to \mathbb{P}\{Y = i\}$ as $n \to \infty$; we write '$X_n \Rightarrow Y$ as $n \to \infty$'.

Recall that if $X_n \sim \mathrm{Bin}(n, p_n)$, and $np_n \to \lambda$ as $n \to \infty$, then $X_n \Rightarrow \mathrm{Po}(\lambda)$ as $n \to \infty$. In other words, if we have n *independent* coin flips with probability $p = p_n$ of heads, where $np_n \to \lambda$ as $n \to \infty$, then the total number of heads converges in distribution to the Poisson distribution with mean λ. We can obtain the same conclusion under weaker assumptions, by looking at the expectation of $(X_n)_r$, the rth factorial moment of X_n. Recall that if $r \in \mathbb{N}$ and $x \geq 0$, we define the *rth factorial moment of x* to be

$$(x)_r = x(x-1)\ldots(x-r+1).$$

We have the following

Theorem 1 (Poisson approximation theorem)**.** *Suppose $X_1, X_2, \ldots$ is a sequence of bounded random variables taking values in $\mathbb{N} \cup \{0\}$. Suppose there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that for any fixed $r \in \mathbb{N}$,*

$$\mathbb{E}[(X_n)_r] \to \lambda^r \quad \textit{as } n \to \infty.$$

Then

$$X_n \Rightarrow \mathrm{Po}(\lambda) \quad \textit{as } n \to \infty.$$

Note that if X_n counts the number of objects of a certain kind in a random structure, then $(X_n)_r$ is simply the number of ordered r-tuples of distinct such objects in the random structure.

To enable us to analyse the joint distribution of $Z_{1,n}$ and $Z_{2,n}$, we will need the following 'joint' version of Theorem 1:

Theorem 2 (Poisson approximation theorem, joint version)**.** *Suppose that for each $i \in \{1, 2, \ldots, m\}$, $X_{i,1}, X_{i,2}, \ldots$ is a sequence of bounded random variables taking values in $\mathbb{N} \cup \{0\}$. Suppose there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}_{\geq 0}$ such that for any fixed $r_1, r_2, \ldots, r_m \in \mathbb{N} \cup \{0\}$,*

$$\mathbb{E}[(X_{1,n})_{r_1}(X_{2,n})_{r_2}\ldots(X_{m,n})_{r_n}] \to \prod_{i=1}^{m} \lambda_i^{r_i} \quad \textit{as } n \to \infty.$$

Then as $n \to \infty$, $(X_{1,n}, X_{2,n}, \ldots, X_{m,n}) \Rightarrow (Y_1, Y_2 \ldots, Y_m)$, where the Y_i's are independent Poisson random variables with $\mathbb{E}[Y_i] = \lambda_i$ for each i.

First, we will prove the following

Lemma 3. *For any $l \in \mathbb{N}$, the expected number of l-cycles in $G^*(n,d)$ satisfies*

$$\mathbb{E}[Z_{l,n}] \to \lambda_l \quad \textit{as } n \to \infty,$$

where $\lambda_l = (d-1)^l/(2l)$.

Proof. Note that l-cycles in $G^*(n,d)$ are in 1-1 correspondence with sets of l edges $\{e_1,\ldots,e_l\}$ of the random matching $\mathcal{M}$, such that there exists a sequence of l distinct vertices $(v_1,\ldots,v_l) \in [n]^l$ with e_i connecting an end emanating from v_i to an end emanating from v_{i+1}, for each i (addition modulo l). Let a_l denote the total number of all such sets of l pairwise disjoint (non-incident) edges in $W^{(2)}$, i.e., the number of sets of l disjoint edges $\{e_1,\ldots,e_l\}$ of $W^{(2)}$ such that there exists a sequence of distinct vertices $(v_1,\ldots,v_l) \in [n]^l$ with e_i connecting an end emanating from v_i to an end emanating from v_{i+1}, for each i. Let b_l denote the number of *sequences* of disjoint edges $(e_1,e_2,\ldots,e_l) \in (W^{(2)})^l$ such that there exists a sequence of l distinct vertices $(v_1,\ldots,v_l) \in [n]^l$ with e_i connecting an end emanating from v_i to an end emanating from v_{i+1}, for each i. Clearly, each *set* corresponds to exactly $2l$ *sequnces* (choose a vertex to start at, and then choose a direction to go in), so

$$2la_l = b_l.$$

Clearly,

$$b_l = (n)_l(d(d-1))^l,$$

so

$$a_l = \frac{(n)_l(d(d-1))^l}{2l}.$$

Note that for any set of l pairwise disjoint (non-incident) edges of $W^{(2)}$, the probability that they all appear in the random matching $\mathcal{M}$ is precisely

$$p_l = \frac{1}{(nd-1)(nd-3)\ldots(nd-2l+1)}.$$

Hence, the expected number of l-cycles in $G^*(n,d)$ satisfies

$$\begin{aligned}
\mathbb{E}[Z_{l,n}] &= a_l p_l \\
&= \frac{(n)_l d(d-1))^l}{2l(nd-1)(nd-3)\ldots(nd-2l+1)} \\
&= (1+O(1/n))\frac{(d-1)^l}{2l} \\
&\to \frac{(d-1)^l}{2l} \quad \text{as } n \to \infty,
\end{aligned}$$

proving the lemma. □

We will now prove the following

Lemma 4. *The expected number of ordered pairs of distinct l-cycles in $G^*(n,d)$ satisfies*

$$\mathbb{E}[(Z_{l,n})_2] \to \lambda_l^2 \quad \textit{as } n \to \infty.$$

Proof. Let $Y = (Z_{l,n})_2$, the number of ordered pairs of distinct l-cycles in $G^*(n,d)$. Write $Y = Y' + Y''$, where Y' is the number of ordered pairs of *vertex-disjoint* l-cycles in $G^*(n,d)$, and Y'' is the number of ordered pairs of distinct l-cycles in $G^*(n,d)$ that are not vertex-disjoint. First, we claim that $\mathbb{E}[Y''] = O(1/n)$. To see this, observe that each unordered pair of non-vertex-disjoint l-cycles corresponds to a copy (in $G^*(n,d)$) of a graph (or multigraph) H that has more edges than vertices. The number of such H's depends only upon l, not on n, so it suffices to prove the following

Claim 1. *If H is a fixed multigraph more edges than (non-isolated) vertices, then the expected number of copies of H in $G^*(n,d)$ is $O(1/n)$.*

Proof of Claim: Let $s = e(H)$, and let $t = |H|$ denote the number of non-isolated vertices of H. Let a_H denote the number of sets of s edges in $W^{(2)}$ which produce copies of H in $G^*(n,d)$. Then $a_H = O(n^t)$, since there are $\binom{n}{t}$ choices for the vertex-set of the copy of H, given this choice, the number of corresponding copies of H depends only upon H and d. The expected number of copies of H in $G^*(n,d)$ is $a_H p_s = O(n^{t-s}) = O(1/n)$, as required. □

It follows that $\mathbb{E}[Y''] = O(1/n)$. It remains to show that the expected number of pairs of vertex-disjoint l-cycles in $G^*(n,d)$ satisfies

$$\mathbb{E}[Y'] \to \lambda_l^2 \quad \text{as } n \to \infty.$$

To see this, observe that the ordered pairs of vertex-disjoint l-cycles in $G^*(n,d)$ are in one-to-one correspondence with ordered pairs

$$(\{e_1, e_2, \ldots, e_l\}, \{f_1, \ldots, f_l\})$$

of sets of k disjoint edges of the random matching $\mathcal{M}$, such that there exists a sequence of distinct vertices,

$$(v_1, \ldots, v_l, w_1, \ldots, w_l) \in [n]^{2l},$$

with e_i connecting an end emanating from v_i to an end emanating from v_{i+1}, for each i, and f_i connecting an end emanating from w_i to an end emanating from w_{i+1}, for each i. Let $a_{l,l}$ denote the total number such ordered pairs of sets of l distinct edges in $W^{(2)}$, i.e. the total number of ordered pairs

$$(\{e_1, e_2, \ldots, e_l\}, \{f_1, \ldots, f_l\})$$

of sets of l disjoint edges of $W^{(2)}$, say $\{e_1, \ldots, e_l\}$ of $W^{(2)}$ such that there exists a sequence of distinct vertices

$$(v_1, \ldots, v_l, w_1, \ldots, w_l) \in [n]^{2l}$$

with e_i connecting an end emanating from v_i to an end emanating from v_{i+1}, for each i, and f_i connecting an end emanating from w_i to an end emanating from w_{i+1}, for each i. Then, arguing as before, we see that

$$a_{l,l} = \frac{(n)_{2l}(d(d-1))^{2l}}{(2l)^2}.$$

The probability that all $2l$ edges in such a pair appear in the random matching $\mathcal{M}$ is p_{2l}, so

$$\begin{aligned}
\mathbb{E}[Y'] &= a_{l,l}p_{2l} \\
&= \frac{(n)_{2l}(d(d-1))^{2l}}{(2l)^2(nd-1)(nd-3)\ldots(nd-4l+1)} \\
&= (1+O(1/n))\left(\frac{(d-1)^l}{2l}\right)^2 \\
&\to \left(\frac{(d-1)^l}{2l}\right)^2 \quad \text{as } n \to \infty,
\end{aligned}$$

proving the lemma. □

In exactly the same way, it can be shown that for any fixed $r_1, r_2, \ldots, r_m \in \mathbb{N} \cup \{0\}$,

$$\mathbb{E}[(Z_{1,n})_{r_1}(Z_{2,n})_{r_2} \ldots (Z_{m,n})_{r_n}] \to \prod_{i=1}^{m} \lambda_i^{r_i} \quad \text{as } n \to \infty,$$

where $\lambda_i = (d-1)^i/(2i)$.

By the Poisson approximation theorem (joint version), it follows that for any fixed m,

$$(Z_{1,n}, \ldots, Z_{m,n}) \Rightarrow (Y_1, \ldots, Y_m) \quad \text{as } n \to \infty,$$

where the Y_i's are independent Poisson random variables with $\mathbb{E}[Y_i] = \lambda_i$ for each i. In particular, $(Z_{1,n}, Z_{2,n}) \Rightarrow (Y_1, Y_2)$, so

$$\begin{aligned}
\mathbb{P}\{G^*(n,d) \text{ is simple}\} &= \mathbb{P}\{(Z_{1,n}, Z_{2,n}) = (0,0)\} \\
&\to \mathbb{P}\{(Y_1, Y_2) = (0,0)\} \\
&= \mathbb{P}\{Y_1 = 0\}\mathbb{P}\{Y_2 = 0\} \\
&= e^{-\lambda_1} e^{-\lambda_2} \\
&= e^{-(d-1)/2} e^{-(d-1)^2/4} \\
&= e^{-(d^2-1)/4}.
\end{aligned}$$

The edge-expansion of random regular graphs

From now on, we will be interested in random d-regular graphs on $[n]$, where $d \geq 3$ is fixed, and n is large. It turns out that if $d \geq 3$, then almost all d-regular graphs on $[n]$ have vertex-connectivity d, and are Hamiltonian (see Appendix). Both statements are false for $d = 1, 2$, which are in a sense 'degenerate' cases. If $d = 1$, and $n \geq 2$ is even, a random 1-regular graph on $[n]$ is simply a uniform random matching of $\{1, 2, \ldots, n\}$, which is always disconnected. If $d = 2$, and $n \geq 3$, then a random 2-regular graph on $[n]$ is simply a uniform random 2-factor of $K_{[n]}$ (a vertex-disjoint union of cycles covering all the vertices), which is almost surely disconnected. To see this, observe that $G^*(n,2)$ is connected if and only if it is an n-cycle. The number $Z_{n,n}$ of n-cycles in $G^*(n,2)$ satisfies

$$\mathbb{E}[Z_{n,n}] = a_n p_n = \frac{n! 2^n}{2n(2n-1)(2n-3)\ldots(1)} = \frac{n! 2^n}{2n(2n-1)!!} = (1+O(1/n))\sqrt{\frac{\pi}{4n}}.$$

Hence, $\mathbb{E}[Z_{n,n}] \to 0$ as $n \to \infty$, so

$$\mathbb{P}\{Z_{n,n} \geq 1\} \leq \mathbb{E}[Z_{n,n}] \to 0.$$

So $G^*(n,2)$ is almost surely disconnected, and therefore so is $G(n,2)$.

We are now ready to prove the following

Theorem 5 (Bollobás, [1]). *Let $d \geq 3$, and let $\eta \in (0,1)$ such that*

$$(1-\eta)\log_2(1-\eta) + (1+\eta)\log_2(1+\eta) > 4/d.$$

Then

$$\mathbb{P}\{h(G(n,d)) \geq (1-\eta)d/2\} \to 1 \quad \text{as } n \to \infty.$$

Proof. Since

$$\mathbb{P}\{G^*(n,d) \text{ is simple}\} \to e^{-(d^2-1)/4} \quad \text{as } n \to \infty,$$

it suffices to prove that almost surely, $h(G^*(n,d)) \geq (1-\eta)d/2$. Hence, from now on, we work in $G^*(n,d)$. Call a set $S \subset [n]$ with $|S| \leq n/2$ 'bad' if $e(S,S^c) < (1-\eta)d|S|/2$. Observe that $e(S,S^c)$ is precisely the number of edges of the random matching of $W = [n] \times [d]$ going between $S \times [d]$ and $S^c \times d$. Let $P(s,k)$ denote the probability that there exists a set $S \subset [n]$ with $|S| = s$, and with exactly k edges of the random matching going between $S \times [d]$ and $S^c \times [d]$. Let k_s be the largest integer less than $(1-\eta)ds/2$ such that $ds - k_s$ is even. By the union bound, we have

$$\mathbb{P}\{h(G(n,d)) < (1-\eta)d/2\} = \sum_{s=1}^{\lfloor n/2 \rfloor} \sum_{\substack{2 \leq k \leq k_s, \\ ds-k \text{ even}}} P(s,k),$$

and

$$P(s,k) \leq \binom{n}{s}\binom{ds}{k}\binom{dn-ds}{k}k!\frac{M(ds-k)M(dn-ds-k)}{M(dn)} := P_0(s,k).$$

Hence, it suffices to show that

$$\sum_{s=1}^{\lfloor n/2 \rfloor} \sum_{\substack{2 \leq k \leq k_s, \\ ds-k \text{ even}}} P_0(s,k) \leq o(1).$$

Note first that if $1 \leq s \leq \lfloor n/2 \rfloor$ and $2 \leq k < k' \leq k_s$, then $P_0(s,k) \leq P_0(s,k')$, so

$$\sum_{s=1}^{\lfloor n/2 \rfloor} \sum_{\substack{2 \leq k \leq k_s, \\ ds-k \text{ even}}} P_0(s,k) \leq \sum_{s=1}^{\lfloor n/2 \rfloor} sP_0(s,k_s).$$

Hence, it suffices to show that

$$\sum_{s=1}^{\lfloor n/2 \rfloor} sP_0(s,k_s) = o(1).$$

To do this, it suffices to show that

1. $P_0(s,k_s) \leq o(1)$ whenever $1 \leq s \leq 100$;
2. $P_0(s,k_s) \leq \epsilon_n/n^2$ whenever $100 \leq s \leq \lfloor n/2 \rfloor$, where $\epsilon_n \to 0$ as $n \to \infty$.

The first statement is easily checked. To prove the second statement, observe that there exists a constant $K_d > 0$ such that

$$P_0(s,k_s) \leq K_d P_0(\lfloor n/2 \rfloor, k_{\lfloor n/2 \rfloor}) \quad \text{whenever } 100 \leq s \leq \lfloor n/2 \rfloor,$$

so it suffices to prove the second statement for $s = \lfloor n/2 \rfloor$. Assume from now on that n is even. (The odd case is completely analogous, but slightly messier.) We have

$$P_0(n/2, k_{n/2}) = \binom{n}{n/2}\binom{dn}{k_s}^2 (k_s)!\frac{(M(dn/2-s))^2}{M(dn)}.$$

Straightforward calculations using the formula (1) give:

$$P_0(n/2, k_{n/2}) \leq C_d \left(2^{4/d}(1-\eta)^{-(1-\eta)}(1+\eta)^{-(1+\eta)}\right)^{n/4d},$$

for some absolute constant $C_d > 0$. Provided $(1-\eta)\log_2(1-\eta) + (1+\eta)\log_2(1+\eta) > 4/d$, the right-hand side is $o(1)$, completing the proof. □

Note that we can take $c_d \to 1/2$ as $d \to \infty$. In some ways, though, the most interesting case of Bollobás' theorem is the $d = 3$ case. It implies that almost surely, $h(G(n,3)) \geq 0.18$. This was historically of great interest. In 1978, Buser had conjectured that for *any* $\epsilon > 0$, for n sufficiently large depending on ϵ, all 3-regular graphs on n-vertices have edge-expansion ratio $< \epsilon$. He disproved this conjecture by constructing, for every even $n \geq 4$, a 3-regular graph on n vertices with edge-expansion ratio at least $1/128$. His construction uses heavy machinery from algebra and geometry. Bollobás' theorem above suppiled the first elementary disproof of Buser's original conjecture (as well as showing it to be massively false, i.e. false for almost all 3-regular graphs).

It would be interesting to determine

$$\psi_d = \sup\{\psi > 0 : \exists \text{arbitrarily large } d\text{-regular graphs } G \text{ with } h(G) \geq \psi d\}.$$

As observed by Bollobás, we always have $\psi_d \leq 1/2$. Indeed, if G is any d-regular graph on $[n]$, and S is a subset of $[n]$ of size $\lfloor n/2 \rfloor$ chosen uniformly at random, then

$$\mathbb{E}[e(S, S^c)] = \frac{dn/2}{\binom{n}{2}} \lfloor n/2 \rfloor \lceil n/2 \rceil = \frac{d}{n-1} \lfloor n/2 \rfloor \lceil n/2 \rceil.$$

Hence,

$$h(G) \leq \frac{d}{n-1} \lceil n/2 \rceil \leq \frac{d(n+1)}{2(n-1)} \to \frac{d}{2} \quad \text{as } n \to \infty,$$

and therefore $\psi_d \leq 1/2$ for all $d \geq 3$.

The reader may be tempted to conjecture that this is roughly the worst possible edge-expansion, but as observed by Bollobás, when $d = 3$, 'balls' have smaller edge-boundary, giving $\psi_3 \leq 1/3$. To see this, let G be a 3-regular graph on $[n]$, and let $k \in \mathbb{N}$ be maximal such that $3 \cdot 2^{k+1} - 4 \leq n$. Pick any $v \in [n]$, $D_i = \{x \in [n] : d_G(x, v) = i\}$ be the set of all vertices of distance (in G) exactly i from v, and let

$$B = \{x \in [n] : d_G(x, v) \leq k\}$$

be the ball of radius k and centre v. Then

$$|B| \leq 1 + 3\sum_{i=0}^{k-1} 2^i \leq 1 + 3(2^k - 1) \leq n/2.$$

Since every vertex in D_k meets an edge from D_{k-1}, it sends at most 2 edges out of B, so we have $e(B, B^c) \leq 2|D_k|$. We now bound $|B|$ from below in terms of $|D_k|$. For $1 \leq i \leq k-1$, every vertex of D_i meets at most two edges from D_{i+1}, and every vertex of D_{i+1} meets at least one edge from D_i. It follows that

$$|D_{i+1}| \leq e(D_i, D_{i+1}) \leq 2|D_i|,$$

so

$$|D_i| \geq \tfrac{1}{2}|D_{i+1}|.$$

Therefore,

$$|B| = 1 + \sum_{i=1}^{k} |D_i| \geq 1 + \left(\sum_{i=1}^{k} 2^{-(k-i)} \right) |D_k| > 2(1 - 2^{-k})|D_k|.$$

Hence,

$$\frac{e(B, B^c)}{|B|} \leq \frac{1}{1 - 2^{-k}} = 1 + O(1/n),$$

and therefore $\psi_3 \leq 1/3$, as claimed.

Appendix 1: The vertex-connectivity of random regular graphs

We will now give a short proof of the following

Theorem 6 (Bollobás / Wormald). *Let $d \geq 3$; then $G(n, d)$ almost surely has vertex-connectivity d.*

Proof. Since every vertex of $G(n, d)$ has degree d, the vertex-connectivity is clearly at most d. It suffices to prove that for any $a_0 \in \mathbb{N}$, almost surely, $G(n, d)$ is such that, for every partition $[n] = A \sqcup S \sqcup B$ such that the removal of S disconnects A from B, and $|A| \leq |B|$, the following conditions hold:

1. If $|A| = 2$, then $|S| \geq 2d - 3$;
2. If $3 \leq |A| \leq a_0$, then $|S| \geq (d - 2)|A|$;
3. If $|A| \geq a_0$, then $|S| \geq (d - 2)a_0$.

First, we observe that for any fixed k, almost surely, $G(n, d)$ is such that every set $T \subset [n]$ with $|T| \leq k$ spans at most $|T|$ edges. Indeed, if there exists a set $T \subset [n]$ with $|T| \leq k$ spanning more than $|T|$ edges, then $G(n, d)$ contains a copy of a graph H of order at most k, and with more edges than vertices. The expected number of copies of such a graph H in $G(n, d)$ is $O(1/n)$, by Claim 1, so almost surely, $G(n, d)$ contains no copy of H.The number of possibilities for H is bounded from above by a function of k alone, so almost surely, $G(n, d)$ contains no copy of any such H.

Now let $s_0 = (d - 2)a_0$, and let $a_1 = 4s_0$. We know that almost surely, $G(n, d)$ is such that every set $T \subset [n]$ with $|T| \leq da_1$ spans at most $|T|$ edges. Suppose from now on that this condition holds. If $|A| = 2$, $[n] = A \sqcup S \sqcup B$, and the removal of S disconnects A and B, then

$$2d - 1 \leq e(A) + e(A, S) \leq e(A \cup S) \leq |S| + 2,$$

so $|S| \geq 2d - 3$. Hence, condition 1 holds.

If $|A| \leq a_1$, $|S| \leq (d - 2)a_1$, $[n] = A \sqcup S \sqcup B$, and the removal of S disconnects A from B, then

$$d|A| = 2e(A) + e(A, S) \leq e(A) + e(A \cup S) \leq |A| + |A \cup S| = 2|A| + |S|,$$

so $|S| \geq (d-2)|A|$. Therefore, conditions 2 holds for all A, and condition 3 holds for all $|A| \leq a_1$. Let $a_2 = \lfloor (n-a_1)/2 \rfloor$; it remains to show that condition 3 holds almost surely for all $a_1 \leq |A| \leq a_2$.

Let $S(a)$ be the probability that condition 3 fails for some $A \subset [n]$ with $|A| = a$. In other words, $S(a)$ is the probability that there exists a 'bad' pair of disjoint sets $A, S \subset [n]$, with $|A| = a$ and $|S| = s_0$, such that there are no edges from A to $[n] \setminus (A \cup S)$. If (A, S) is 'bad', then the random matching M matches all the ends emanating from A to ends emanating from $A \cup S$. The probability that this occurs is at most

$$\left(\frac{a+s_0}{n}\right)^{ad/2}.$$

The total number of possibilites for (A, S) is $\binom{n}{s_0}\binom{n-s_0}{a}$, so, using the union bound,

$$\begin{aligned}
S(a) &= \mathbb{P}\left(\bigcup_{(A,S)} \{(A,S) \text{ is bad}\}\right) \\
&\leq \sum_{A,S} \mathbb{P}\{(A,S) \text{ is bad}\} \\
&\leq \binom{n}{s_0}\binom{n-s_0}{a}\left(\frac{a+s_0}{n}\right)^{ad/2} \\
&\leq n^{s_0}\binom{n-s_0}{a}\left(\frac{a+s_0}{n}\right)^{ad/2} \\
&=: S'(a)
\end{aligned}$$

Observe that $S'(a_1) = O(n^{s_0+a_1-3a_1/2}) = O(n^{s_0-a_1/2}) = O(n^{-s_0}) = o(1)$, and for $a_1 \leq a \leq a_2 - 1$, we have $S'(a+1)/S'(a) \leq O(n^{-1/2})$. It follows that

$$\sum_{a=a_1}^{a_2} S(a) \leq \sum_{a=a_1}^{a_2} S'(a) = o(1),$$

so by the union bound, condition 3 holds almost surely for all $a_1 \leq |A| \leq a_2$, as required. □

Bollobás conjectured in 1981 that for $d \geq 3$, almost all d-regular graphs on $[n]$ are Hamiltonian. This turned out to be much harder; it was eventually proved by Robinson and Wormald [5] in 1994. As one might expect, if $\mathcal{A}$ is a monotone-increasing property of graphs (meaning that it is closed under the addition of edges), then for any integers $d \leq d'$, $G(n,d) \in \mathcal{A}$ almost surely implies that $G(n,d') \in \mathcal{A}$ almost surely, and similarly, $G^*(n,d) \in \mathcal{A}$ almost surely implies that $G^*(n,d')$ almost surely. Hence, Bollobás' conjecture reduces to the statement that almost all 3-regular graphs on $[n]$ are Hamiltonian. The reader is encouraged to consult [4] for a proof.

Appendix 2: Other models of random regular graphs

For some applications, other models of random regular graphs are more useful. One such is the *permutation model* of random $2d$-regular graphs. This produces a random (labelled) $2d$-regular graph on $[n]$ as follows. Start with our n vertices, $\{1, 2, \ldots, n\}$. Pick d permutations $\sigma_1, \ldots, \sigma_d$ uniformly at random from the symmetric group S_n, with replacement. For each $l \in [d]$, and each $i \in [n]$, join i to $\sigma_l(i)$ (forming a loop if $\sigma_l(i) = i$). This produces a random $(2d)$-regular *multigraph*, which we denote $R^*(n, 2d)$: in general, it may have loops and multiple edges. (As before, a loop contributes 2 to the degree of a vertex.) The multigraph is simple if and only if

1. None of the permutations have any fixed points;
2. None of the permutations have any 2-cycles (when written in disjoint cycle notation);
3. No two permutations agree anywhere.

As before, we produce a random $(2d)$-regular graph $R(n, 2d)$ by conditioning on the event that $R^*(n, 2d)$ is simple. It is relatively easy to prove that

$$\mathbb{P}\{R^*(n, 2d) \text{ is simple}\} \to e^{-d} e^{-d/2} e^{-\binom{d}{2}} = e^{-d^2/2-d} \quad \text{as } n \to \infty,$$

i.e. the probability of this event is bounded from below by a positive constant depending only upon d.

To generate a random $(2d+1)$-regular graph, when n is even, we can add to $R^*(n, 2d)$ the edges of a uniform random matching of $[n]$. We let $R^*(n, 2d+1)$ denote the resulting $(2d+1)$-regular multigraph, and we produce a random $(2d+1)$-regular graph by conditioning on the event that $R^*(n, 2d+1)$ is simple. Again, the probability that $R^*(n, 2d+1)$ is a simple graph is bounded from below by a positive constant depending only upon d.

The distribution of $R(n, d)$ is different to that of $G(n, d)$, even for n large: there are properties $\mathcal{A}$ with $\lim_{n\to\infty} \mathbb{P}\{R(n, d) \in \mathcal{A}\} \neq \lim_{n\to\infty} \mathbb{P}\{G(n, d) \in \mathcal{A}\}$. However, it was recently proved by Greenhill, Janson, Kim and Wormald [3] that the two models are *contiguous*, meaning that any event that occurs almost surely in one model occurs almost surely in the other.

For some properties $\mathcal{A}$, it is easier to show that $\mathcal{A}$ holds almost surely in the permutation model, and then deduce the same statement in the uniform model by contiguity, than to work directly with $G(n, 2d)$ or $G^*(n, 2d)$ via the configuration model. This is often true of *spectral* properties (properties of the eigenvalues of the adjacency matrix). As will be discussed later in the course, Friedman was able to use the permutation model to show that almost surely, the eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of the adjacency matrix of $G(n, d)$ satisfy

$$\max\{|\lambda_2|, |\lambda_n|\} \leq 2\sqrt{2d-1} + o(1).$$

For d-regular graphs G, the 'second spectral modulus' $\nu(G) = \max\{|\lambda_2|, |\lambda_n|\}$ controls several important properties of the graph G. As we will see, it determines how evenly the edges of G are distributed, and it also determines the convergence rate of the simple symmetric random walk on G. A d-regular graph with $\nu(G) \leq 2\sqrt{d-1}$ is called a *Ramanujan graph*. As mentioned in Lecture 1, the construction of arbitrarily large d-regular Ramanujan graphs (for d fixed) was one of the most important problems in theoretical computer science. Friedman's result says that almost all d-regular graphs on $[n]$ are 'almost Ramanujan'.

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