Cascade Dynamics of Multiplex Propagation

Damon Centola,[1, 2, *] Michael W. Macy,[2] and Víctor M. Eguíluz[3, †]

[1]*Department of Sociology, Columbia University, New York, NY 10027*
[2]*Department of Sociology, Cornell University, Ithaca, NY 14853[‡]*
[3]*Instituto Mediterráneo de Estudios Avanzados IMEDEA (CSIC-UIB), E-07122 Palma de Mallorca, Spain*
(Dated: February 2, 2008)

Random links between otherwise distant nodes can greatly facilitate the propagation of disease or information, provided contagion can be transmitted by a single active node. However we show that when the propagation requires simultaneous exposure to multiple sources of activation, called *multiplex propagation*, the effect of random links is just the opposite: it makes the propagation more difficult to achieve. We calculate analytical and numerically critical points for a threshold model in several classes of complex networks, including an empirical social network.

PACS numbers: 87.23.Ge, 89.65.-s, 89.75.Hc, 87.19.Xx

Introduction.– Recently much attention has been paid to complex networks as the skeleton of complex systems [1, 2, 3, 4, 5]. For example, recent advances in complex systems have shown that most real networks display the small world property: they are as clustered as a regular lattice but with an average path length similar to a random network [1]. More precisely, it has been shown that surprisingly few bridge links are needed to give even highly clustered networks the "degrees of separation" characteristic of a "small world". Interestingly, these random links significantly increase the rate of propagation of contagions such as disease and information [1, 6, 7]. For *simple propagation* –such as the spread of information or disease– in which a single active node is sufficient to trigger the activation of its neighbors, random links connecting otherwise distant nodes achieve dramatic gains in propagation rates by creating "shortcuts" across the graph [8, 9]. Sociologists have long argued that bridge links between disjoint neighborhoods promote the diffusion of information and innovation, a regularity known as the "strength of weak links" [8].

In addition to simple propagation, *multiplex propagation*, in which node activation requires simultaneous exposure to multiple active neighbors, is also common in the social world. Fads, stock market herds, lynch mobs, riots, grass roots movements, and environmental campaigns (such as curb side recycling) share the important property that a bystander's probability of joining increases with the level of local participation by her neighbors [10]. In this case, one neighbor acting alone is rarely sufficient to trigger a cascade. These cascades often display a second important property: they typically unfold in clustered networks. Empirical studies have consistently found that recruitment to social movements is most effective in locally dense networks characterized by

strong interpersonal ties [11, 12]. Short cycles expose potential recruits to multiple and overlapping influences that provide the strong social support required to motivate costly investments of time, effort, and resources.

In this paper, we use a threshold model to analyze the effect of bridge ties in complex networks on the dynamics of multiplex propagation. [10, 13, 14]. Our results show that contrary to the results for cascades in small worlds networks [1, 6, 7], for multiplex propagation, random links to distant nodes reduce propagation rates. Moreover, too many random links can prevent cascades from occurring altogether. To test the results of our model, we examine its predictions on an empirical network with scale-free degree distribution.

The threshold model.– The system is composed of a set of N agents located at the nodes of a network. Each agent can be in one of two states: 1 indicates that the agent is active, otherwise its state is 0. Each agent is characterized by a fixed threshold $0 \leq T \leq 1$. The dynamics are defined as follows. Each time t a node i is selected at random. Then

1. if its state is 1 (active), then it will remain active;

2. however, if its state is 0, then it becomes active, changing its state to 1, if only if the fraction of its neighbors in the active state is equal to or larger than T.

In order to isolate the effect of the network topology from the effect of threshold distribution, we assign every node an identical threshold T, which determines the fraction of neighbors required to activate it. By definition, a single active seed is insufficient for multiplex propagation. Hence, we seed the network by randomly selecting a focal node and activating this node and all of its neighbors. For any graph, there is a critical threshold T_c above which propagation is not possible.

Critical Thresholds in Regular and Random Graphs.– First, we compare critical thresholds in random and regular networks with identical size N and average degree $\langle k \rangle$.
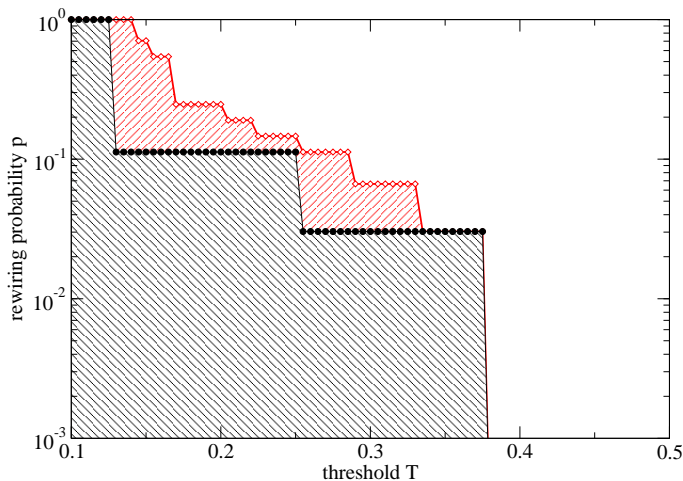
FIG. 1: Cascade window for small-world networks. Results are averaged over 1000 realizations in a 10,000 node network. The shaded area indicates when cascades occur for small-world networks obtained after rewiring (red diamonds) or permuting (black circles) links.
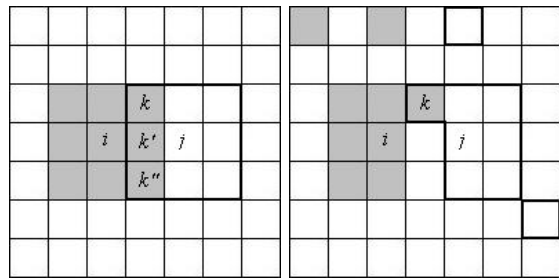

FIG. 2: Bridges between neighborhoods in (left) a regular lattice and (right) after some rewiring. Rewiring deceases the bridge width between i and j making cascade propagation more difficult.

As defined by the cascade condition in Ref. [13], for a random graph of size N in which all the nodes have the same degree $\langle k \rangle \ll N$ and the same threshold T, as N approaches infinity the probability that two nodes in the initial seed neighborhood will have a common neighbor approaches zero. Thus, the critical threshold for a random graph is approximated by

$$T_c^r = \frac{1}{\langle k \rangle} \; , \tag{1}$$

which corresponds to the limiting case of simple propagation, and shows that multiplex propagation cannot succeed on sparse random graphs.

The critical threshold for a regular one-dimensional lattice is [9]

$$T_c^{1d} = \frac{1}{2} \; . \tag{2}$$

While in a one-dimensional ring with average degree $\langle k \rangle$ the critical threshold is independent of the interaction length [Eq. (2)], in a random graph with the same average degree $\langle k \rangle$ the critical threshold decreases with $\langle k \rangle$ [Eq. (1)]. Thus, the difference between the critical thresholds of regular one-dimensional lattices and random networks increases with the average degree $\langle k \rangle$, making the one-dimensional lattice much more vulnerable to multiplex propagation than an equivalent random network.

This feature is also observed in two-dimensional lattices. In a two-dimensional lattice with near and next-nearest neighbors (also called a Moore neighborhood) the critical threshold is [9]

$$T_c^{2dnn} = \frac{3}{8} = 0.375 \; . \tag{3}$$

As the interaction length in the two-dimensional lattice increases, the critical threshold approaches the upper limit of $1/2$ [9]. Thus, increasing $\langle k \rangle$ increases the differences in the critical thresholds between regular and random networks, making clustered regular networks able to support comparatively greater amounts of multiplex propagation than random networks.

Small-world networks.– We next explore the transition in critical thresholds that occurs in the small-world regime between perfect regularity and pure randomness by randomizing links in a two-dimensional regular lattice with nearest and next-nearest neighbors. We study the effects of bridge ties on the success of multiplex propagation using two different perturbation algorithms. One is the usual *rewiring* technique [1]: each link is broken with probability p and reconnected to a randomly selected node. We then observed the likelihood of successful cascades, as p increases from 0 to 1, repeating the experiment for different threshold values. The second algorithm rewires links in such a way that nodes keep their degrees (and thus the original degree distribution is conserved) by *permuting* links [15]: a link connecting nodes i and j is permuted with a link connecting nodes k and l. For both cases, a cascade is *successful* if it reaches at least 90% of network nodes.

For $T > T_c^{2dnn} = 3/8$ (the critical threshold for $p = 0$), cascades are precluded for all p. Permuting links such that all nodes have the same degree $k = 8$, if $T < 1/8$ (the critical value for $p = 1$), cascades are guaranteed for all p. Thus, for multiplex propagation randomization is only meaningful within the window $1/8 \leq T \leq 3/8$. Figure 1 reports the phase diagram for cascade frequency for thresholds in this range, as the original regular neighborhoods ($\langle k \rangle = 8$) are randomized with probability $0.001 \leq p \leq 1$. Despite small differences between the two algorithm used for the perturbation of the network, the phase diagram shows that cascades are bounded above by $T_c = 3/8$ and below by $T_c = 1/8$. As thresholds are increased, the critical value of p decreases, making cascades less likely in the small-world network region of the phase space.

Figure 2 shows the effects of perturbation on two neighborhoods with the focal nodes i and j. i's neighborhood
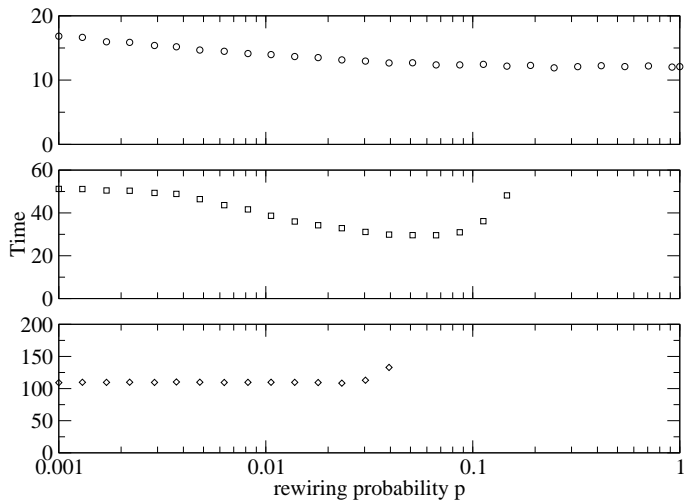
FIG. 3: Time to activate all the nodes ($N = 10000$) for an initial seed for $T = 0.12$, 0.24, and 0.36. Time has been averaged over 100 realizations. The randomized networks have been obtained using the permutation algorithm.

is a seed neighborhood (shaded) and j's neighborhood (outlined) is inactive. In Figure 2a, the nodes k, k', and k'' are shared by both neighborhoods i and j. By acting as bridges between the two neighborhoods, these nodes allow multiplex propagation to spread from one to the other. Random rewiring reduces the width of the bridge between the neighborhoods by reducing the common neighbors shared by i and j, as shown in Figure 2b, where random rewiring has eliminated two of the common neighbors of i and j. In the resulting network, i's neighborhood can only activate j through k; thus, if j requires multiple sources of activation, i's neighborhood will no longer be sufficient to activate j.

When ties are randomly rewired, local changes to neighborhood structure dramatically affects the dynamics of propagation. Fig. 4 shows the different growth rates of cascades in regular and rewired networks. In a regular lattice the growth of active nodes follows a power law with an exponent around 2, due to the two dimensional nature of the network. However in the small network the growth initially follows an exponential law and then it rapidly expands and activates all the nodes.

In Fig. 3 we show the average time required for the initial seed to reach the full population for different values of the threshold and the rewiring probability p, using the permutation algorithm. For simple propagation, random perturbation of ties reduces propagation time as expected. However, perturbing the network lowers its critical threshold, thus reducing the viability of contagions that spread by multiplex propagation. As shown in Fig. 3, at the critical point T_c, rates for multiplex propagation diverge, increasing to infinity with increasing p. Thus, although random links can increase the rate of propagation, they can also preclude propagation by lowering the critical threshold of the network.
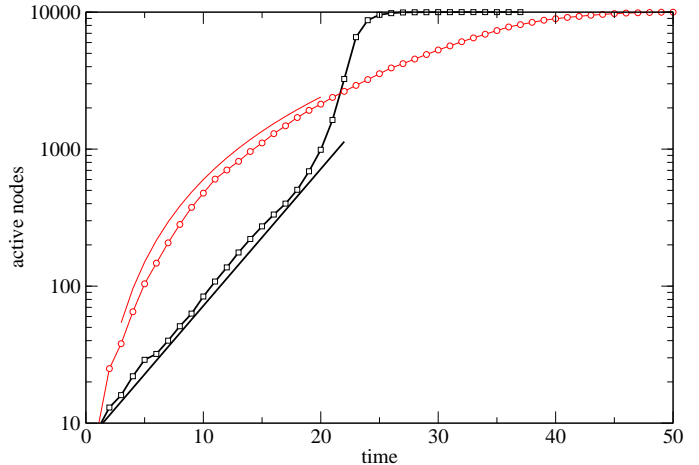

FIG. 4: Total number of active nodes as a function of time from an initial seed of 9 active neighbor nodes and $T = 0.24$ for one realization. While for (circles) the regular lattice ($p = 0$) the curve follows a power law growth (solid line increases as $(time)^2$), (squares) for a small-world network (following the permutation algorithm with $p = 0.1$) the growth is initially exponential (solid line).

Empirical scale-free networks.- Regular lattices are an important theoretical demonstration of multiplex propagation because they can have very wide bridges between near-neighbors. Nevertheless, except for special cases where spatial patterns of interaction dominate the structure of the network of interaction [16], regular lattices are not a good representation of real networks. We therefore extended our analysis to an empirical social network. In particular, we consider the Internet Movie Data Base as an illustrative example. Figure 5 reports the average relative size of cascades, $\langle s \rangle$, in the Internet movie database (IMDB). It is worth noting that due to the nature of the transition for cascade behavior, the average frequency of cascades is equivalent to the average relative size of cascades, $\langle s \rangle$. The black line shows $\langle s \rangle$ for the original network ($p = 0$), where $T_c \simeq 0.1$, while the red line represents the randomized IMDB ($p = 1$). In the randomized graph, $T_c \simeq 0.04$ (approximately $1/\langle k \rangle$ for estimated $\langle k \rangle = 25$). Consistent with our results for regular lattices, the socially clustered IMDB network supports multiplex propagation that cannot propagate on the randomized network.

Conclusions.– Using a threshold model, we have analyzed simple and multiplex propagation in different classes of complex networks. The relevant bridging mechanism for multiplex propagation is not the dyadic link but multiple short paths between source and target. As a regular lattice is randomized, there are fewer common neighbors to provide multiple simultaneous sources of activation. Thus, while networks with long range links have been shown to promote simple propagation in small-
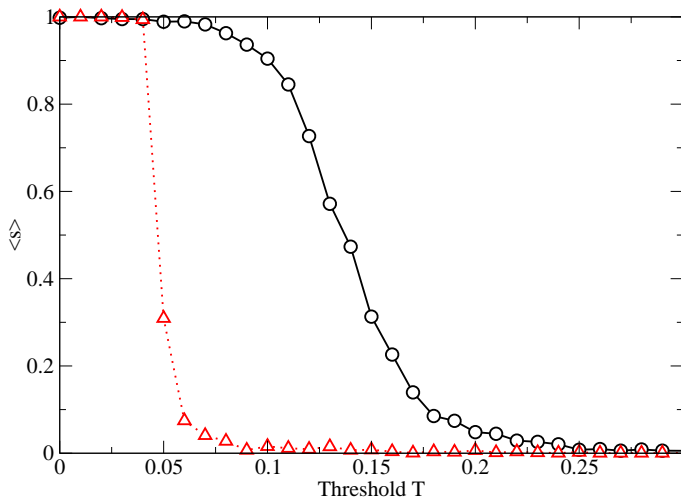
FIG. 5: Effect of perturbation on multiplex propagation in the IMDB network. For the unperturbed network (black circles), $T_c \simeq 0.1$. For the randomized network (red triangles), $T_c \simeq 0.04$ (approximately $1/z$ for $\langle k \rangle \simeq 25$). The perturbed network cannot support multiplex propagation that is possible on the real structured social network. The randomized network has been obtained permuting links in order to keep the original degree distribution [15].

world networks, they inhibit multiplex propagation. This implies that random links do not promote diffusion if the credibility of information or the willingness to adopt an innovation depends on receiving independent confirmation from multiple sources.

The qualitative differences between multiplex and simple propagation caution about extrapolating from the spread of disease or information to the spread of participation in political, religious, or cultural movements. These movements may not benefit from "the strength of weak links" and may even be hampered by processes of global integration. More broadly, many of the important empirical studies of the effects of small-world networks on the dynamics of cascades may need to take into account the possibility that propagation may be multiplex. In fact, for the dynamics of multiplex propagation, our results highlight the inhibitory effects of networks typically thought to be advantageous for cascades.

[1] D.J. Watts, S.H. Strogatz, Nature **393**, 440 (1998).
[2] S.H. Strogatz, Nature **410**, 268 (2001).
[3] R. Albert A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[4] S.N. Dorogovtsev, J.F.F. Mendes, Adv. Phys. **51**, 1079 (2002).
[5] M.E.J. Newman, SIAM Review **45**, 167-256 (2003).
[6] D.J. Watts, Am J. Soc. **105**, 493 (1999).
[7] M.E.J. Newman, J. Stat. Phys. **101**, 819 (2000).
[8] M. Granovetter, Am. J. Soc. **78**, 1360 (1973).
[9] S. Morris, Rev. Econ. Studies **67**, 57 (2000).
[10] M. Granovetter, Am. J. Soc. **83**, 1420 (1978).
[11] D. McAdam, Am. J. Soc. **92**, 64 (1986).
[12] D. McAdam and R. Paulsen, Am. J. Soc. **99**, 640 (1993)
[13] D.J. Watts, Proc. Natl. Acad. Sci. USA **99**, 5766 (2002).
[14] P.S. Dodds, D.J. Watts, Phys. Rev. Lett. **92**, 218701 (2004).
[15] S. Maslov, K. Sneppen, and U. Alon, in *Handbook of Graphs and Networks*, edited by S. Bornholdt and H.G. Schuster (Wiley-VCH and Co., Weinheim, 2003).
[16] R.V. Gould, *Insurgent Identities: Class, Community, and Protest in Paris from 1848 to the Commune*. Chicago: The University of Chicago Press (1995).