

Analyzing Structural and Symmetrical Properties of C. Elegans Neural Network

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I. INTRODUCTION

The study of neuroscience could be dated back to the Edwin Smith Surgical Papyrus, written in the 17th century BC, which contains the earliest recorded reference to the brain. The hieroglyph for brain, occurring eight times in the papyrus, describes the symptoms, diagnosis, and prognosis of two patients, wounded in the head, who had compound fractures of the skull[1]. In the last two centuries, thanks to the development of biological technologies, researchers can finally probe into real neural networks of various animals, which greatly benefits the study of neuroscience. Understanding the mechanism humans' brain works on from the aspect of neural network is interesting but challenging due to the complexity of network structure. Hence, scientists have been instead studying behaviors of creatures with a simple neural network. In 1970s, biologist Sydney Brenner and his colleagues began preserving tiny hermaphroditic roundworms known as *Caenorhabditis elegans*(C. elegans) in agar and osmium fixative, slicing up their bodies like pepperoni and photographing their cells through a powerful electron microscope. Their goal was to create a wiring diagram a map of all 302 neurons in the C. elegans nervous system as well as all the 7,000 connections between those neurons, which is named as connectome today. In 1986 the scientists published a draft of the diagram. More than 20 years later, Dmitri Chklovskii of Janelia Farm Research Campus and his collaborators published a even more comprehensive version.

Currently, *Caenorhabditis elegans* is the only creature that we have complete knowledge of its nervous system. Detailed knowledge, as posted on WormAtlas, has accumulated information about many aspects of this system, including the anatomical location, lineage information, and functional role of each neuron[2]. There has been a growing interest in the network properties of the C. elegans nervous system since connectome data was made available. Researches on C. elegans include basic network properties analysis of neural networks[3], locomotion understanding and simulation[4], ways to track fluctuating stimuli[5]. One of the most famous project with respect to C. elegans is the openworm project[6], which is an international open science project to simulate the C. elegans at the cellular level as a simulation. Openworm has a long-term goal of modeling all 959 cells of the C. elegans, with the first stage to model the worm's locomotion by simulating the 302 neurons and 95 muscle cells.

Another interesting topic of C. elegans neural network is symmetry detection. Although the overall architecture of the nervous system of most animals shows a bilateral symmetry, evidence shows that there are patterns of left-right asymmetry in the brains of some species. Left-right asymmetry appears in different ways including size, gene expression or function. Connectome data of C. elegans offers a unique opportunity to address how symmetrical neuronal assemblies deviate to create functional lateralizations. Although symmetry study of C. elegans network is a promising topic, we found few works concentrating on this issue. Thus, we focused mainly on this topic, and performed experiments and thorough analysis on the symmetry of C. elegans neural network from the aspect of a graph.

In our work, instead of studying C. elegans' behaviors and biological properties, we focused on exploiting network analysis algorithms and techniques to get a deeper understanding of the worm's neural network structure, and explain the network properties in biological ways. Our work consists of three parts. First, properties of C. elegans neural network were analyzed. We computed basic statistics such as node degree distribution, diameter, graph density, clustering coefficient etc. Furthermore, more advanced algorithms were performed to detect communities and examine PageRank scores, as well as authority and hub scores. Second, we tried to model C. elegans neural network using random networks. In specific, Erdos-Renyi networks and preferential attachment model networks were generated and compared to the C. elegans neural network. It was found that the C. elegans neural network shows remarkable difference from random networks and is well structured, potentially has hierarchy and consists of multiple functional groups. Third, we fully studied the symmetry of C. elegans neural network from the aspect of a graph, instead of location or function. Experiments implied that although global symmetry is not found, C. elegans neural network possesses a well-defined local symmetry pattern.

The following of this paper is organized as follows: in Section II, related works to our project is briefly addressed. In Section III, we discussed the dataset and resources we use through the project. Section IV is focused on network property analysis including degree distribution, clustering coefficient, etc. on C. elegans neural network. Symmetry detection is thoroughly narrated in Section V, and future work and conclusion is covered in Section VI.

II. RELATED WORKS

Most researches on *C. elegans* neural network focus on the worm's behavioral patterns and biological properties. Therefore, we have studied papers addressing a wide range of topics—some are focused on biology and neuroscience, others are on network analysis algorithms.

A. *C. Elegans Worm and Neuroscience Basics*

Neural networks are considered to be composed of smaller circuits serving specific functions[7]. For example, in crustaceans, a network of less than 30 neurons coordinates chewing and propulsion of the food[8], [9], [10]. There is a similar concept in cognitive psychology, that our brains are divided into different regions, and each region is in charge of a specific set of related tasks. The human processor model, for example, takes computers as an analogy to human brains, and consists of three processors, perceptual, motor, and cognitive processor, and two memory storage places—long-term memory and working memory[11]. With these concepts in mind, we would expect the *C. elegans* neural network to display explicit clustering phenomena, which is a major focus of our work.

One of the earliest papers on *C. elegans* neural network found one interesting property of the network—triangular patterns of connectivity. "One of the striking features of the connectivity diagrams is the high incidence of triangular connections linking three classes." [12] Triangular patterns of connectivity is a property of social networks, where it is more likely for one person to build relationship with a friend of his or her friends than with strangers. In the social settings, this property is linked with the notion of communities. Therefore, triangular patterns of connectivity again points to strong clustering phenomena.

B. *Network Analysis on C. Elegans Neural Network*

As *C. elegans* is the only creature that has complete connectivity data (connectome), there are multiple papers analyzing its network structural properties. In paper *The Structure of the Nervous System of the Nematode Caenorhabditis elegans*, White et al. introduced the connectivity data of *C. elegans*[12]. "The hermaphrodite nervous system has a total complement of 302 neurons, which are arranged in an essentially invariant structure." [12] They also grouped the neurons according to their morphologies and connectivities, into 118 classes[12]. This paper analyzed the network primarily from a biological perspective and did not involve much network analysis techniques.

Later, researches on the topological and network properties of the *C. elegans*' neural network came to be a heated topic, aiming at understanding the connectivity from a network analysis perspective. In *Structural Properties of the Caenorhabditis elegans neural Network*, Lav R. Varshney et al. calculated statistical and topological properties of the network, such as degree distributions, synaptic multiplicities, and small-world properties[13]. They also identified neurons that may play central roles in information processing, and network motifs that could serve as functional modules of the network[13]. Based on those statistical properties, they found similarities between the *C. elegans* neural network and mammalian neocortex, suggesting more general principles of neural networks[13]. *Topological Cluster Analysis Reveals the Systemic Organization of the Caenorhabditis elegans Connectome*, however, focuses more on the modular organization of the network in addition to the topological properties[14]. Yunkyun Sohn et al. used a modularity-based community detection method for directed, weighted networks to examine hierarchically organized modules in the complete wiring diagram (connectome) of *C. elegans* and to investigate their topological properties[14]. They identified anatomical clusters in the *C. elegans* connectome, and the hierarchical organization of the five clusters explains the systemic cooperation that occurs among the structurally segregated biological circuits to produce higher-order complex behaviors[14]. Moreover, in paper *The Rich Club of the C. elegans neural Connectome*, Emma K. Towilson et al. used graph theory to investigate the neural connectome of *C. elegans*[15]. They identified a small number of highly connected neurons as a rich club interconnected with high efficiency and high connection distance[15]. They describe rich club neurons as connector hubs, with high betweenness centrality, and many intermodular connections to nodes in different modules[15]. They compare the rich club organization in a cellular connectome to human neuroimaging on much larger scale networks, suggesting that this may be a general and scale-invariant principle of brain network organization[15].

Beyond researches on *C. elegans* neural network, network connectivity analysis also applies on much larger scale neural system. In *Complex network measures of brain connectivity: Uses and interpretations*, Rubinov and Sporns used network analysis techniques to characterize brain networks with a small number of neurologically meaningful and easily computable measures[16]. More precisely, they described measures that can detect functional integration and segregation, and quantified centrality of individual brain regions or pathways[16].

C. *Network Symmetry Analysis*

There are some previous works done on the symmetry of neural networks from a biological view. Oliver Hobert et al.'s work[17] studied the left-right asymmetry in *C. elegans*' nervous system, but the concept of symmetry is defined by position (on the *C. elegans*). More specifically, connection information was ignored in their work. Christian Pohl's work[18] studied Left-right patterning in the *C. elegans* embryo, including establishing and maintaining directional L-R asymmetry, Wnt/Planar cell polarity pathways and L-R asymmetry etc. Furthermore, Tao Sun et al.'s studied brain asymmetry and handedness through some molecular approaches[19].

In our work, we studied the symmetry from the aspect of graph, taking connection into consideration but ignoring the position information. Some of our concepts are derived from[20], which proposed the idea of local symmetry and discussed the condition of local and global symmetry appearing in Erdos-Renyi network. Some other similar works include the study of structure entropy based on automorphism partition[21] and the development of a novel approach to canonical labeling where symmetries are found first and then used to speed up the canonical labeling algorithms[22].

III. DATASET

In our work, we majorly used the dataset *Neural Network of a Caenorhabditis Elegans Worm* from the course website, which is a directed, weighted graph containing 297 neurons and 2345 neuron links with weights (available at[23]). The problem of this dataset is that we do not know the meaning of each neuron, which makes it hard for interpretation. Thus, in addition to the dataset from the 224W course website, we use the neuron wiring data provided on WormAtlas[2], which contains neuron name and connection type.

IV. NETWORK PROPERTY ANALYSIS

A. Generate Random Networks

We began with comparing C. elegans neural network to random networks such as Erdos-Renyi networks[24] and preferential attachment model networks[25]. By comparison, we can get more insights of the C. elegans network.

100 random networks were generated from each method described below. For both of these two methods, we fixed the number of nodes to be 300 and the number of edges to be 2400, making it comparable to the C. elegans neural network (which has 297 nodes and 2345 edges).

Erdos-Renyi. An Erdos-Renyi network $G(n, p)$ with parameters n, p is constructed by generating n nodes first and then for each of the $\frac{n(n-1)}{2}$ potential edges, choosing randomly and independently with probability p [24].

Preferential Attachment Model. A Preferential Attachment graph (network) is generated by the following steps: one new node is created at each time tick; each potential out-edge starting from the new node is born with probability proportional to the destination node's in-degree plus a default bias. In other words, new nodes prefer to attach to well-connected nodes over less-well connected nodes[25].

B. Visualization

C. elegans neural network is a relatively small one, containing only about 300 nodes, making it possible and reasonable to be visualized. Color map visualization with various statistics could bring us valuable information on the network structure. We used Gephi[26] to visualize these networks following the *Force-atlas graph layout algorithm*[27].

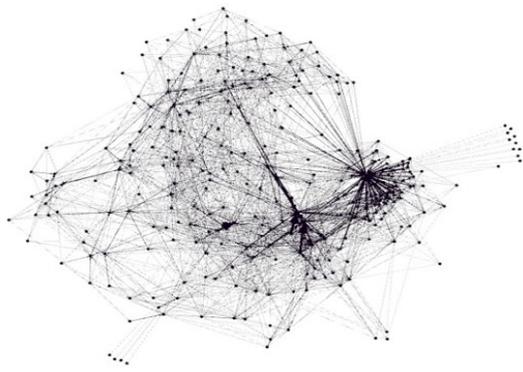
Fig. 1 (a) is the layout of C. elegans neural network and (b) is the layout of a random network generated by preferential attachment model. From the visualization, we see that the structure of C. elegans neural network differs a lot from the one generated by the preferential attachment model. In addition, after we colored the nodes according to their in-degrees and out-degrees (Fig. 1 (c), (d)), we see there is one "super" node with a very high in-degree and multiple nodes with high out-degree. However the node with a very high in-degree does not have a high out-degrees. One possible interpretation is that the C. elegans neural system collects information from multiple sensory neurons, gathers and processes information in "hubs" or "centers" of the system and then passes the processed information to the remaining system.

C. Network Properties

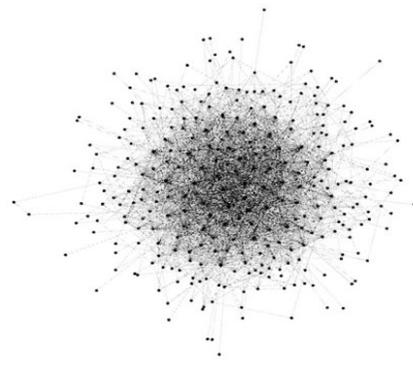
In this section, we present network properties of C. elegans neural network, and compare which to the properties of random networks. For random networks, the statistics are the mean of all 100 samples. Comparison results are shown in Table I.

Basic Properties. Average degree, network diameter, graph density, average path length.

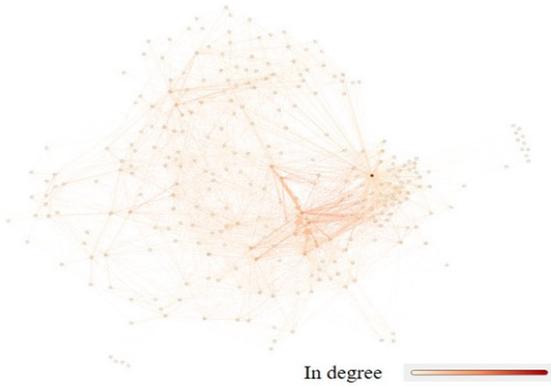
Degree Distribution. The degree distribution is shown in Fig. 2. Note that for Erdos-Renyi and preferential attachment model, we randomly picked one sample among 100 samples and used it in the degree distribution plot.



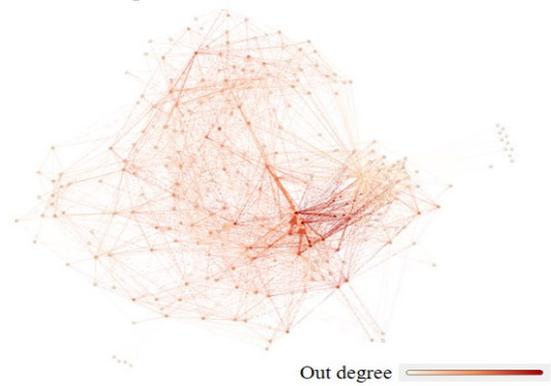
(a) Layout of *C. elegans* neural network



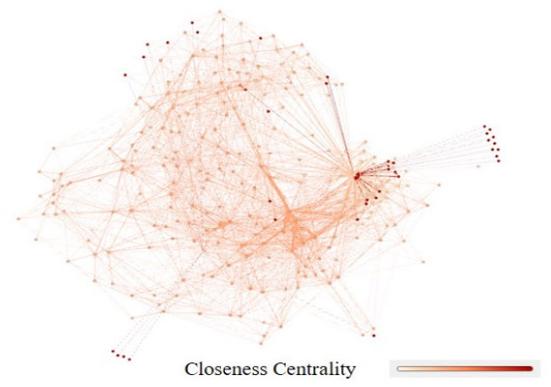
(b) Layout of a random network following preferential attachment model



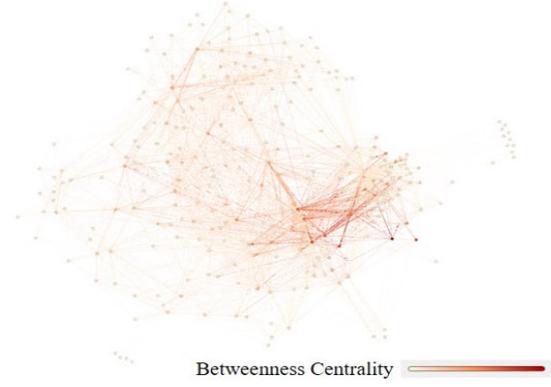
(c) In-degree of *C. elegans* neural network



(d) Out-degree of *C. elegans* neural network



(e) Closeness centrality of *C. elegans* neural network



(f) betweenness centrality of *C. elegans* neural network



(g) Authority score of *C. elegans* neural network



(h) Clustering coefficient of *C. elegans* neural network

Fig. 1: Network visualizations

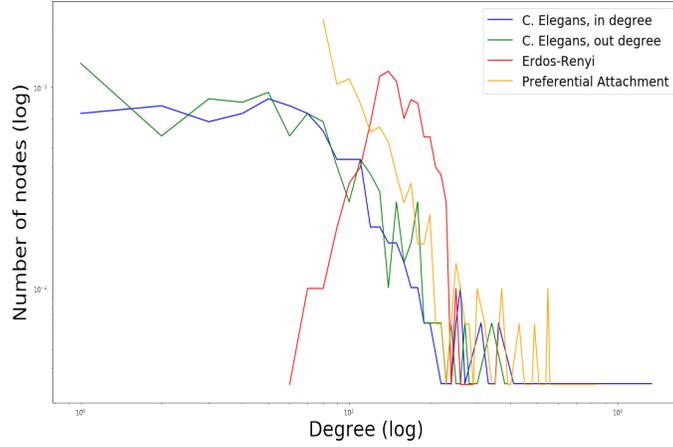


Fig. 2: Degree distribution

Clustering Coefficient. The clustering coefficient is defined as

$$c_n = \frac{t_n}{d_n(d_n - 1)} \quad (1)$$

where t_n is the number of triangle around node n and d_n is the degree of node n [3]. We calculate clustering coefficient for each individual node and take the average as one measure of the network. The network visualization with clustering coefficient color map is in Fig. 1 (h).

Transitivity. One problem of average clustering coefficient is that nodes with low degrees may influence it disproportional[16]. The transitivity measure does not suffer this problem, and is defined as

$$\text{Transitivity} = \frac{\sum_{n \in N} 2t_n}{\sum_{n \in N} d_n(d_n - 1)} \quad (2)$$

Closeness Centrality. The closeness centrality measures the average distance from a given starting node to all other nodes in the network. Defined as

$$c_i = \frac{1}{\sum_{j \in \text{Ner}(i)} w_{ij}}, \quad (3)$$

where j is in neighbor of node i and w_{ij} is the weight of the edge between node j and node i [28]. The closeness centrality is calculated for each node.

The network visualization with closeness centrality color map is in Fig. 1 (e). Surprisingly we see that several nodes with only one edge to the "super" in-degree node have very high closeness centrality. This may be because the "super" in-degree node can reach other nodes easily, leading to nodes directly has edge to it also share high closeness centrality.

Betweenness Centrality. The betweenness centrality measures how often a node appears on a shortest paths between nodes in the network. Defined as

$$c_k = \sum_{i \neq j \neq k} \frac{\sigma_{i,j}(k)}{\sigma_{i,j}}, \quad (4)$$

where $\sigma_{i,j}$ is the number of shortest path among node i and node j and $\sigma_{i,j}(k)$ is the number of shortest path among node i and node j through node k [29]. The betweenness centrality is calculated for each node. The network visualization with betweenness centrality color map is in Fig. 1 (f).

PageRank Page estimates how important one node is in the network in term of information flow, defines as

$$R_n = \frac{1-d}{N} + d \sum_{i \in \text{Ner}(n)} \frac{R_i A_{n,i}}{\sum_j A_{n,j}}, \quad (5)$$

where d is the damping factor, $\text{Ner}(n)$ denotes the neighbor of node n , and A is the weighted adjacency matrix[30]. The PageRank score will be calculated for each node and we will use the largest value as one measure of the network.

TABLE I: Network Properties Comparison

	Average Degree	Network Diameter	Graph Density	Average Path Length	Clustering Coefficient
C. Elegans	7.896	14	0.027	3.992	0.169
Erdos-Renyi	8	5.10	0.027	2.951	0.027
Preferential Attachment	8.867	7.12	0.031	2.361	0.165
	Max PageRank	Max Authority	Max Hub	Efficiency	Modularity
C. Elegans	0.125	0.307	0.214	0.445	0.484
Erdos-Renyi	0.0066	0.145	0.134	0.457	0.21
Preferential Attachment	0.013	0.129	0.352	0.466	0.142

Authority and Hubs. Authorities \mathbf{a} and hubs \mathbf{h} are defined as

$$\begin{cases} \mathbf{a} = \alpha \mathbf{A} \mathbf{h} \\ \mathbf{h} = \beta \mathbf{A}^T \mathbf{a}, \end{cases} \quad (6)$$

where \mathbf{A} is the adjacency matrix and $\lambda = (\alpha\beta)^{-1}$ is the largest eigenvalue of $\mathbf{A}\mathbf{A}^T$ [31]. The Authority and Hub scores is calculated for each node and we use the largest values (from each) as two measures of the network. The network visualization with authority color map is in Fig. 1 (g).

Efficiency. The efficiency of a network measures how well and efficiently information can be transmitted among the network, defined as

$$E_i = \frac{\sum_{j \neq i} d_{ij}^{-1}}{N - 1}, \quad (7)$$

where d_{ij} is the shortest path from node i to node j [16]. The efficiency is calculated for each node and we use the average value among all nodes as one measure of the network.

Modularity. Modularity is one measure of communities in the network, defined as

$$Q = \sum_{u \in M} [e_{uu} - (\sum_{v \in M} e_{uv})^2], \quad (8)$$

where the network is fully subdivided into a set of non-overlapping modules M , and e_{uv} is the portion of all edges that connect nodes in module u with nodes in module v [32]. Note that we treat the network as undirected network when calculating the modularity.

D. Discussion

The C. elegans neural network is much better structured than the random networks, which is shown directly from the network visualization (Fig. 1), and confirmed by various network properties as well. There are several interesting observations: i) the network diameter of C. elegans neural network is much larger than that of random networks while its efficiency is comparable to others. This indicates that the overall network structure of C. elegans is highly efficient even with some very long paths. We have expected a few long paths because neurons are located throughout the body of C. elegans. ii) The maximum PageRank of C. elegans neural network is much larger than random networks, reconciling with the visualization showing that C. elegans neural network has "super" nodes. iii) The modularity of C. elegans neural network is significantly larger than that of the random networks, together with the larger PageRank, may indicate that the C. elegans network is not only highly structured, but also has hierarchy and consists of multiple functional group coordinated by some "high-level" neuron nodes.

V. SYMMETRY

A. Global Symmetry

The abstract concept of symmetry is defined in terms of isomorphisms and automorphisms. In this section, we define these notions for general graphs, whether directed or undirected. We also include some useful properties and observations that are used during our analysis on C. elegans neural network.

Definition 1. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a function $f : G_1 \rightarrow G_2$ that is bijective, mapping nodes to nodes and edges to edges. Equivalently, G_1 and G_2 are isomorphic if there is a bijective function $f : V_1 \rightarrow V_2$ such that $(u_1, v_1) \in E_1$ if and only if $(f(u_1), f(v_1)) \in E_2$.

This is a known equivalence relation on the set of all graphs and can be thought of as identifying graph structures, ignoring graph representations. We can think of isomorphic graphs as relabeling of each other. Another wording for isomorphism is structurally equivalent. One should be careful that two graphs are the same only if the labelings are exactly the same. An isomorphism of a graph with itself is called automorphism.

Definition 2. Given a graph G , $f : G \rightarrow G$ is a graph automorphism if f is an graph isomorphism.

Every graph has a trivial automorphism, namely the identity function. Graphs with nontrivial automorphisms are symmetric. Graphs with only trivial automorphisms are asymmetric. Automorphisms of graphs have many useful properties and reveal a lot about graph structures. For example, the set of graph automorphisms has a group structure, so the set of graph automorphisms on a graph defines equivalence classes in its vertex set. Vertices in the same equivalence class can be seen as "structurally indistinguishable". These interpretations lead us to the following definitions about symmetry.

Definition 3. Given a graph $G = (V, E)$. Two vertices $v_1, v_2 \in V$ are globally symmetric if there is an automorphism $f : G \rightarrow G$ such that $f(v_1) = v_2$. Furthermore, for a graph G , if there exists a symmetric pair $v_1 \neq v_2 \in G$, then we say G is globally symmetric.

Remark. G is globally symmetric if and only if it has a nontrivial automorphism.

Proposition 1. Given a graph G and its adjacency matrix A . The following statements are equivalent:

- 1) G is globally symmetric
- 2) There exists a non-trivial row-switching transformation matrix R such that $R^{-1}AR = A$.

Proof: 1) \Leftrightarrow 2) An isomorphism $f_R : G \rightarrow G'$ corresponds to a permutation on V with appropriate operations on the adjacency matrix determined by the vertex set permutation. In fact, $f_R(A)$ is a change of basis transformation of A : $f_R(A) = R^{-1}AR$. To understand why R must be a row-switching transformation matrix, we decompose the relabeling process into swaps.

Any permutation can be decomposed into multiple swaps, or 2-cycles. $f_R = f_{R_{i_1j_1}} \circ f_{R_{i_2j_2}} \circ \dots \circ f_{R_{i_kj_k}}$. A labeling swap between two nodes v_i, v_j corresponds to two swaps in the adjacency matrix A : a swap between rows i, j in A and a swap between columns i, j in A . Write it in matrix notation, $f_{R_{ij}}(A) = R_{ij}^{-1}AR_{ij}$, where R_{ij} is the row-switching transformation matrix corresponding to a swap of nodes i, j . Therefore, $f_{R_{ij}}(A) = (R_{i_1j_1}R_{i_2j_2} \dots R_{i_kj_k})^{-1}A(R_{i_1j_1}R_{i_2j_2} \dots R_{i_kj_k}) = R^{-1}AR$. $R = R_{i_1j_1}R_{i_2j_2} \dots R_{i_kj_k}$ is a row-switching transformation matrix.

An automorphism f is an isomorphism from G to G , therefore, f is an automorphism on G if and only if $f(A) = A$. So G is globally symmetric if and only if there exists a non-trivial row-switching transformation matrix R such that $R^{-1}AR = A$. ■

There are many kinds of global symmetries. By group theory, automorphisms in G can have any order that is less than or equal to $|V|$. Here, f has order d means $f^d = Id$ and $f^k \neq Id$ for all $0 < k < d$. For example, the identity automorphism has order 1.

In our study of the *C. elegans* neural network, we are particularly interested in the left-right structural symmetry. In left-right symmetries, there is a central axis made of some nodes and all other nodes has a counterpart node. We can formulate left-right symmetries in the following way.

Definition 4. Given a graph $G = (V, E)$, $f : G \rightarrow G$ is a left-right symmetry of G if f is a non-trivial automorphism of G and $f = f^{-1}$. Furthermore, we say $v \in V$ is on the symmetric axis if v is a fixed point of f on G , that is $f(v) = v$.

Remark. f is a left-right symmetry of G if f as an element in the automorphism group has order 2.

B. Approximation Symmetry

Global symmetries can be easily disrupted by noises. One way to get symmetry information that's stable under noises is to have a notion of local symmetry. We define k -neighborhoods first and then define what it means by k -locally symmetric.

Definition 5. Given a graph $G = (V, E)$. Define a vertex set $S \subseteq V$, an open k -neighborhood of S in G : $\mathcal{N}_G^k(S) = \{v \in V : d(v, S) \leq k\}$. A closed k -neighborhood of S in G , $\mathcal{N}_G^k[S]$, is the subgraph of G induced by $\mathcal{N}_G^k(S)$.

Remark. In this paper, we sometimes refer to closed k -neighborhoods as k -neighborhoods.

Definition 6. Given a graph $G = (V, E)$. Two vertices $v_1, v_2 \subseteq V$, are k -locally symmetric if there is an isomorphism f between $\mathcal{N}_G^k[\{v_1\}]$ and $\mathcal{N}_G^k[\{v_2\}]$ such that $f(v_1) = v_2$.

Proposition 2. Given a graph $G = (V, E)$, $k \in \mathbb{N}$, and two vertices $v_1, v_2 \subseteq V$. Then:

- 1) If v_1, v_2 are $(k+1)$ -locally symmetric, then v_1, v_2 are k -locally symmetric.
- 2) If $k \geq \text{diam}(G)$, then v_1, v_2 are k -locally symmetric if and only if v_1, v_2 are globally symmetric.

For practical reasons, given a graph G and a number k , we would like to find out k -locally symmetric pairs on G . There is an algorithmic method to achieve this[22]. However, we have found out a purely algebraic way to figure this out.

Theorem 1. In graph $G = (V, E)$, $v_1, v_2 \in V$, and denote the adjacency matrices for $\mathcal{N}_G^k[\{v_1\}]$ and $\mathcal{N}_G^k[\{v_2\}]$ as A_1 and A_2 , both preserving the order of nodes as in the adjacency matrix of G . Then the following statements are equivalent.

- 1) v_1, v_2 are k -locally symmetric.
- 2) There exists some a non-trivial row-switching transformation matrix R such that $R^{-1}A_1R = A_2$.

Proof: By definition, v_1, v_2 are k -locally symmetric if and only if there is a graph isomorphism $f_R : \mathcal{N}_G^k[\{v_1\}] \rightarrow \mathcal{N}_G^k[\{v_2\}]$ such that $f(v_1) = v_2$. This isomorphism f_R can be written as $f_R(A_1) = R^{-1}A_2R$ for some non-trivial row-switching transformation matrix R . The proof is essentially the same as in the proof of Proposition 1. ■

Remark. If $\mathcal{N}_G^k[\{v_1\}] \neq \mathcal{N}_G^k[\{v_2\}]$, A_1 being similar to A_2 is equivalent to saying A_1 and A_2 have the same Jordan forms. In particular, if both A_1 and A_2 are diagonalizable, then k -local symmetry corresponds to A_1 and A_2 having the same set of eigenvalues.

Remark. If $\mathcal{N}_G^k[\{v_1\}] = \mathcal{N}_G^k[\{v_2\}]$, then v_1, v_2 being k -locally symmetric is equivalent to v_1, v_2 being globally symmetric in $\mathcal{N}_G^k[\{v_1\}]$. Indeed, in this case, Proposition 1 collides with Theorem 1.

Proposition 3. Given a graph $G = (V, E)$, and any two vertices $v_1, v_2 \subseteq V$, if the adjacency matrices for $\mathcal{N}_G^1[\{v_1\}]$ and $\mathcal{N}_G^1[\{v_2\}]$ are similar (follow the definition in Definition 6), then v_1 and v_2 are 1-locally symmetric.

Proof: Assume that we have $\mathcal{N}_G^1[\{v_1\}]$ and $\mathcal{N}_G^1[\{v_2\}]$ are similar, there are two possible situations:

$\mathcal{N}_G^1[\{v_1\}]$ and $\mathcal{N}_G^1[\{v_2\}]$ are two different graphs. In this case, according to Proposition 2, v_1 and v_2 are 1-locally symmetry.

$\mathcal{N}_G^1[\{v_1\}]$ and $\mathcal{N}_G^1[\{v_2\}]$ are the same graph. In this case, we can denote the neighbor of v_1 as $\{v_2, S\}$, and the neighbor of v_2 as $\{v_1, S\}$. One real example is shown in Fig. 5. Apparently, if we swap the label of v_1 and v_2 , we will get the same adjacency matrix. Thus, we claim that v_1 and v_2 are symmetric. ■

C. Experiment

Experiments were performed on the C. elegans neural network to detect local and global symmetry. Shown by numerical result, we do not have global symmetry in C. elegans neural network, but local symmetry does exist. Details of local symmetry is shown in Fig. 3 and Table. II.

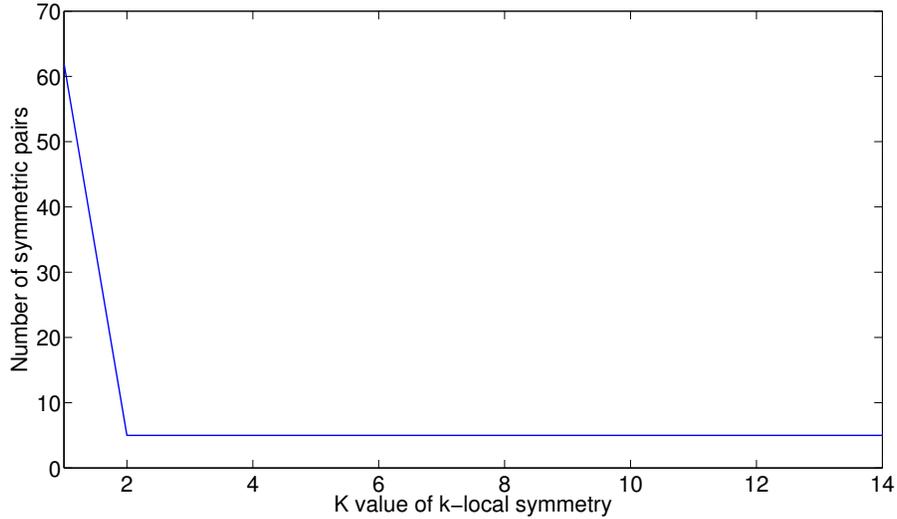


Fig. 3: Number of symmetric pairs versus k value of k-local symmetry

It is shown in Fig. 3 that when the k value of k -local symmetry increases, the number of symmetric pairs will first decrease and maintain stable. The reason for maintaining stable is that the cardinality of $\mathcal{N}_G^k(v)$ for v having a symmetric twin at $k - 1$ keeps unchanged after some certain k , as shown in Fig. 4.

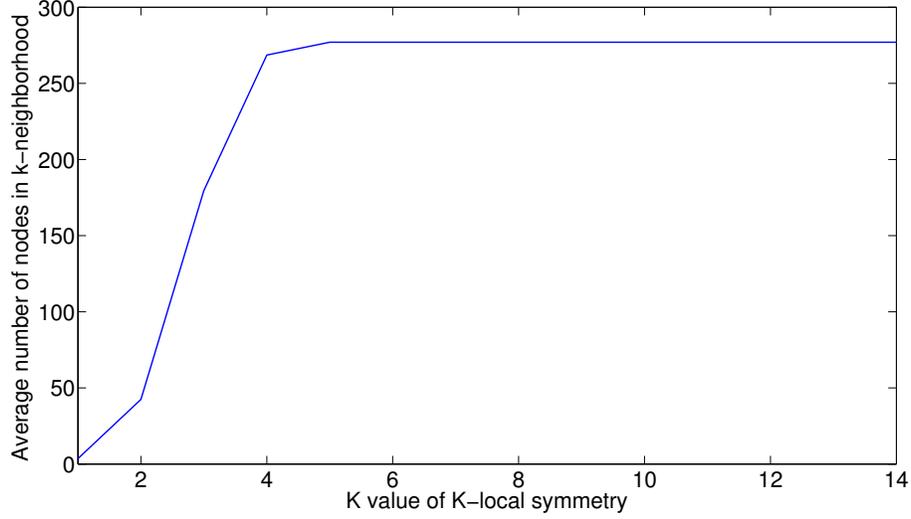


Fig. 4: Average number of cardinality of $N_G^k(v)$ versus k value of k-local symmetry

TABLE II: Statistics of k-local symmetry

k value of k-local symmetry	number of symmetry pairs	some examples
0	N/A	N/A
1	62	{RMHL, RMHR}, {DB05, DB06}, {SIBDR, VD08}
2	5	{AS08, AS10}, {AS08, VA10}, {AS10, VA10}
3	5	{AS08, AS10}, {AS08, VA10}, {AS10, VA10}
≥ 4	5	{AS08, AS10}, {AS08, VA10}, {AS10, VA10}

As shown by Table. II, neuron AS08, AS10, VA10 are symmetric twin to each other. In fact, each of AS08, AS10, VA10 is connected only to AVAL and AVAR. Thus, these 3 neurons are symmetric to each other. Visualization is shown in Fig. 5.

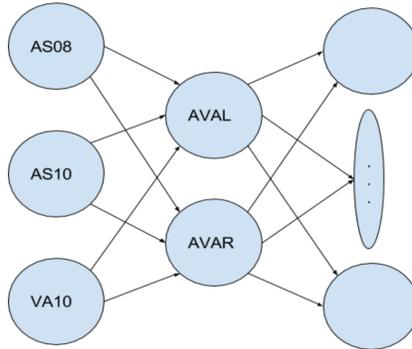


Fig. 5: one example of local symmetry

VI. CONCLUSION

In this paper, we analyzed the structural properties of *C. elegans* neural network and compared them with random networks. We found *C. elegans* neural network is well structured and well organized. Furthermore, we analyzed symmetrical property on the *C. elegans* neural network based on purely structural information. We formulated the notion of global symmetry as left-right symmetry, which is a specific kind of symmetry, and found out that symmetry is vulnerable to even very small perturbations or changes on the network. In order to do useful analysis on the network, we relied on local symmetry analysis method. We have derived equivalence conditions using matrix operations for local symmetry. Using the derived matrix condition, we found that local symmetry exists only for small neighborhoods, and many of these neighborhoods have special neighborhood structures which encourage localized neighborhood symmetries. Since there are two thirds of neuron pairs that are left-right symmetric to

each other (in terms of functions and locations) with the ventral nerve cord central to the *C. elegans* body[17], we think our result implies that a lot of interruptions to symmetry happen close to the center of the *C. elegans* body. This finding is consistent with the finding that neural structures on the left or right side of the brain in vertebrates and invertebrates. The asymmetry in left-right brains may be required for generating or retrieving long-term memory[33]. Thus, we guess that the structural asymmetry we have found is crucial to the *C. elegans* worm's core functions including memory. We leave this guess here for future works.

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