Small world networks

CS 224W
Outline

- Small world phenomenon
  - Milgram’s small world experiment

- Local structure
  - clustering coefficient
  - motifs

- Small world network models:
  - Watts & Strogatz (clustering & short paths)
  - Kleinberg (geographical)
  - Kleinberg, Watts/Dodds/Newman (hierarchical)

- Small world networks: why do they arise?
Small world phenomenon: Milgram’s experiment
**Milgram’s experiment**

**Instructions:**
Given a target individual (stockbroker in Boston), pass the message to a person you correspond with who is “closest” to the target.

**Outcome:**

20% of initiated chains reached target  
average chain length = 6.5

- “Six degrees of separation”
email experiment

- 18 targets
- 13 different countries
- 60,000+ participants
- 24,163 message chains
- 384 reached their targets
- average path length 4.0

Interpreting Milgram’s experiment

- Is 6 a surprising number?
  - In the 1960s? Today? Why?

- Pool and Kochen in (1978 established that the average person has between 500 and 1500 acquaintances)
Ignore for the time being the fact that many of your friends’ friends are your friends as well. If everyone has 500 friends, the average person would have how many friends of friends?

- 500
- 1,000
- 5,000
- 250,000
Quiz Q:

With an average degree of 500, a node in a random network would have this many friends-of-friends-of-friends (3\textsuperscript{rd} degree neighbors):

- 5,000
- 500,000
- 1,000,000
- 125,000,000
Interpreting Milgram’s experiment

- Is 6 is a **surprising** number?
  - In the 1960s? Today? Why?

- If social networks were random… ?
  - Pool and Kochen (1978) - ~500-1500 acquaintances/person
  - ~ 500 choices 1st link
  - ~ 500² = 250,000 potential 2nd degree neighbors
  - ~ 500³ = 125,000,000 potential 3rd degree neighbors

- If networks are completely cliquish?
  - all my friends’ friends are my friends
  - what would happen?
If the network were completely cliquish, that is all of your friends of friends were also directly your friends, what would be true:

- (a) None of your friendship edges would be part of a triangle (closed triad)
- (b) It would be impossible to reach any node outside the clique by following directed edges
- (c) Your shortest path to your friends’ friends would be 2
If all your friends of friends were also your friends, you would be part of an isolated clique.
Uncompleted chains and distance

- Is 6 an accurate number?

- What bias is introduced by uncompleted chains?
  - are longer or shorter chains more likely to be completed?
Attrition

![Graph showing attrition in the spread of messages through a social network chain. The graph plots the probability of passing on a message against the position in the chain, with data points indicating the average and 95% confidence interval.]

Source: An Experimental Study of Search in Global Social Networks: Peter Sheridan Dodds, Roby Muhamad, and Duncan J. Watts (8 August 2003); Science 301 (5634), 827.
Quiz Q:

If each intermediate person in the chain has 0.5 probability of passing the letter on, what is the likelihood of a chain being completed:

- of length 2?
- of length 5?

- sends for sure
- receives

Chain of length 2

Passes on with probability 0.5
Quiz Q:

if each intermediate person in the chain has 0.5 probability of passing the letter on, what is the likelihood of a chain of length 5 being completed

(a) $\frac{1}{2}$
(b) $\frac{1}{4}$
(c) $\frac{1}{8}$
(d) $\frac{1}{16}$
Estimating the true distance

Source: An Experimental Study of Search in Global Social Networks: Peter Sheridan Dodds, Roby Muhamad, and Duncan J. Watts (8 August 2003); Science 301 (5634), 827.
Is 6 an accurate number?

Do people find the shortest paths?

Killworth, McCarty, Bernard, & House (2005):
less than optimal choice for next link in chain is made ½ of the time
What does it mean to be 1, 2, 3 hops apart on Facebook, Twitter, LinkedIn, Google Plus?
Transitivity:
- if A is connected to B and B is connected to C what is the probability that A is connected to C?
- my friends’ friends are likely to be my friends
Global clustering coefficient

$$C = \frac{3 \times \text{number of triangles in the graph}}{\text{number of connected triples of vertices}}$$
Local clustering coefficient (Watts & Strogatz 1998)

- For a vertex $i$
  - The fraction pairs of neighbors of the node that are themselves connected
  - Let $n_i$ be the number of neighbors of vertex $i$

$$C_i = \frac{\text{# of connections between i’s neighbors}}{\text{max # of possible connections between i’s neighbors}}$$

$$C_i \text{ directed} = \frac{\text{# directed connections between i’s neighbors}}{n_i \times (n_i - 1)}$$

$$C_i \text{ undirected} = \frac{\text{# undirected connections between i’s neighbors}}{n_i \times (n_i - 1)/2}$$
Local clustering coefficient (Watts & Strogatz 1998)

- Average over all $n$ vertices

$$C = \frac{1}{n} \sum_{i} C_i$$

$n_i = 4$
- Max number of connections: $4 \times 3/2 = 6$
- 3 connections present

$$C_i = 3/6 = 0.5$$

link present
- link absent
The clustering coefficient for vertex $i$ is:

(a) 0
(b) $1/3$
(c) $1/2$
(d) $2/3$
\( n_i = 3 \)

- there are 2 connections present out of max of 3 possible

\( C_i = \frac{2}{3} \)
Small world phenomenon:

- high clustering
- low average shortest path

\[ C_{\text{network}} \gg C_{\text{random graph}} \]

\[ l_{\text{network}} \approx \ln(N) \]

What other networks can you think of with these characteristics?
### Comparison with “random graph” used to determine whether real-world network is “small world”

<table>
<thead>
<tr>
<th>Network</th>
<th>size</th>
<th>av. shortest path</th>
<th>Shortest path in fitted random graph</th>
<th>Clustering (averaged over vertices)</th>
<th>Clustering in random graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Film actors</td>
<td>225,226</td>
<td>3.65</td>
<td>2.99</td>
<td>0.79</td>
<td>0.00027</td>
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<td>MEDLINE co-authorship</td>
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<td>0.56</td>
<td>1.8 x 10^{-4}</td>
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<tr>
<td>E.Coli substrate graph</td>
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<td>2.9</td>
<td>3.04</td>
<td>0.32</td>
<td>0.026</td>
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<td>C.Elegans</td>
<td>282</td>
<td>2.65</td>
<td>2.25</td>
<td>0.28</td>
<td>0.05</td>
</tr>
</tbody>
</table>
Small world phenomenon: Watts/Strogatz model

Reconciling two observations:

- High clustering: my friends’ friends tend to be my friends
- Short average paths

Watts-Strogatz model:
Generating small world graphs

As in many network generating algorithms
- Disallow self-edges
- Disallow multiple edges

Select a fraction $p$ of edges
Reposition on of their endpoints

Add a fraction $p$ of additional edges leaving underlying lattice intact

Each node has $K \geq 4$ nearest neighbors (local)

tunable: vary the probability $p$ of rewiring any given edge

small $p$: regular lattice

large $p$: classical random graph
Quiz question:

Which of the following is a result of a higher rewiring probability?

(a) Left  (b) Right  (c) insufficient information
What happens in between?

- Small shortest path means low clustering?
- Large shortest path means high clustering?

Through numerical simulation

- As we increase $p$ from 0 to 1
  - Fast decrease of mean distance
  - Slow decrease in clustering
Clust coeff. and ASP as rewiring increases

1% of links rewired

10% of links rewired

Trying this with NetLogo

http://web.stanford.edu/class/cs224w/NetLogo/SmallWorldWS.nlogo
The probability that a connected triple stays connected after rewiring:
- Probability that none of the 3 edges were rewired \((1-p)^3\)
- Probability that edges were rewired back to each other very small, can ignore

\[
\text{Clustering coefficient} = C(p) = C(p=0) \times (1-p)^3
\]

Which of the following is a description matching a small-world network?

(a) Its average shortest path is close to that of an Erdos-Renyi graph
(b) It has many closed triads
(c) It has a high clustering coefficient
(d) It has a short average path length
WS Model: What’s missing?

- Long range links not as likely as short range ones
- Hierarchical structure / groups
- Hubs
“The geographic movement of the [message] from Nebraska to Massachusetts is striking. There is a progressive closing in on the target area as each new person is added to the chain”

S. Milgram ‘The small world problem’, Psychology Today 1,61,1967
nodes are placed on a lattice and connect to nearest neighbors

additional links placed with

\[ p(\text{link between } u \text{ and } v) = (\text{distance}(u,v))^{-r} \]

how does the probability of long-range links affect search?

http://web.stanford.edu/class/cs224w/NetLogo/SmallWorldSearch.nlogo
When \( r=0 \), links are randomly distributed, \( \text{ASP} \sim \log(n) \), \( n \) size of grid

When \( r=0 \), any decentralized algorithm is at least \( a_0 n^{2/3} \)

\[ p \sim p_0 \]

When \( r<2 \), expected time at least \( \alpha_r n^{(2-r)/3} \)
Overly localized links on a lattice

When $r > 2$ expected search time $\sim N^{(r-2)/(r-1)}$

$p \sim \frac{1}{d^4}$
When $r=2$, expected time of a DA is at most $C \left( \log N \right)^2$

$p \sim \frac{1}{d^2}$
$\lambda^2 |R| < |R'| < \lambda |R|$

$k = c \log^2 n$

calculate probability that $s$ fails to have a link in $R'$
Quiz Q:

What is true about a network where the probability of a tie falls off as distance^{-2}

(a) Large networks cannot be navigated
(b) A simple greedy strategy (pass the message to the neighbor who is closest to the target) is sufficient
(c) There are fewer long range ties than short range ones
(d) If the number of nodes doubles, the average shortest path will be twice as long
Origins of small worlds: group affiliations

Social distance—Bipartite networks:
Hierarchical small-world models: Kleinberg

Hierarchical network models:
Individuals classified into a hierarchy,
\( h_{ij} = \text{height of the least common ancestor} \).

\[
p_{ij} : b^{-\alpha h_{ij}}
\]

e.g. state-county-city-neighborhood
industry-corporation-division-group

Group structure models:
Individuals belong to nested groups
\( q = \text{size of smallest group that } v,w \text{ belong to} \)

\[
f(q) \sim q^{-\alpha}
\]

Watts, Dodds, Newman (Science, 2001) individuals belong to hierarchically nested groups

multiple independent hierarchies $h=1,2,...,H$ coexist corresponding to occupation, geography, hobbies, religion...

Navigability and search strategy: Reverse small world experiment

Killworth & Bernard (1978):
- Given hypothetical targets (name, occupation, location, hobbies, religion…) participants choose an acquaintance for each target
- Based on (most often) occupation, geography
- Only 7% because they “know a lot of people”
- Simple greedy algorithm: most similar acquaintance
- Two-step strategy rare

Navigability and search strategy: Small world experiment @ Columbia

Successful chains disproportionately used
• weak ties (Granovetter)
• professional ties (34% vs. 13%)
• ties originating at work/college
• target's work (65% vs. 40%)

... and disproportionately avoided
• hubs (8% vs. 1%) (+ no evidence of funnels)
• family/friendship ties (60% vs. 83%)

Strategy: Geography -> Work
Search in power-law networks

Motivation
Power-law (PL) networks, social and P2P

Analysis of scaling of search strategies in PL networks

Simulation
artificial power-law topologies, real Gnutella networks
How do we search?

Mary

Who could introduce me to Richard Gere?

Bob

Jane

Richard Gere
AT&T Call Graph

# of telephone numbers from which calls were made

# of telephone numbers called

Aiello et al. STOC '00
Gnutella network

power-law link distribution

proportion of nodes vs. number of neighbors

- data
- power-law fit \( \tau = 2.07 \)

summer 2000,
data provided by Clip2
Preferential attachment model

Nodes join at different times

The more connections a node has, the more likely it is to acquire new connections

Growth process produces power-law network
file sharing w/o a central index

queries broadcast to every node within radius \textit{ttl}

$\Rightarrow$ as network grows, encounter a bandwidth barrier (dial up modems cannot keep up with query traffic, fragmenting the network)

Clip 2 report
Gnutella: To the Bandwidth Barrier and Beyond
http://www.clip2.com/gnutella.html#q17
power-law graph

number of nodes found
Poisson graph

number of nodes found

93
19
15
11
7
3
1
Search with knowledge of 2nd neighbors
Outline of search strategy

pass query onto only one neighbor at each step

OPTIONS

requires that nodes sign query
  - avoid passing message onto a node twice

requires knowledge of one’s neighbors degree
  - pass to the highest degree node

requires knowledge of one’s neighbors neighbors
  - route to 2nd degree neighbors
Generating functions

- "Random graphs with arbitrary degree distributions and their applications", PRE, cond-mat/0007235

- Generating functions for degree distributions

\[ G_0(x) = \sum_{k=0}^{\infty} p_k x^k \]

- Useful for computing moments of degree distribution,
- component sizes, and average path lengths
Introducing cutoffs

\[ k_{\text{max}} < N - 1 \] a node cannot have more connections than there are other nodes

This is important for exponents close to 2

\[
\sum_{1}^{\infty} p_k = \sum_{1}^{\infty} C_\tau \frac{1}{x^\tau} = 1 \quad C_2 = \frac{6}{\pi^2}
\]

\[ p(k > 1000, \tau = 2) = \sum_{1000}^{\infty} p_k \sim 0.001 \]

Probability that none of the nodes in a 1,000 node graph has 1000 or more neighbors:

\[ (1 - p(k > 1000, \tau = 2))^{1000} \sim 0.36 \]

without a cutoff, for \( \tau = 2 \) have > 50% chance of observing a node with more neighbors than there are nodes

for \( \tau = 2.1 \), have a 25% chance
Selecting from a variety of cutoffs

1. $k_{\text{max}} < N$

2. $p_k = Ck^{-\tau}e^{-k/\kappa}$  
   Newman et al.

3. $p_k = \begin{cases} 
Ck^{-\tau} & k < (CN)^{\tau} \\
0 & \text{otherwise} 
\end{cases}$  
   Aiello et al.

Generating Function

$$G_0(x) = C \sum_{k=1}^{(CN)^{\tau}} k^{-\tau} x^k$$

1 million websites (~1997)

- Plot showing proportion of sites with so many links against the square root of $N$.
Aiello’s ‘conservative’ vs. Havlin’s ‘natural’ cutoff

\[ N \cdot p_k = 1 \]

\[ Ck^{-\tau} = N^{-1} \]

\[ k \sim N^{\frac{1}{\tau}} \]

\[ N \cdot \sum_{k=k_{\max}}^{\infty} p_k = 1 \]

\[ \int_{k=k_{\max}}^{\infty} c k^{-\tau} \sim N^{-1} \]

\[ k_{\max}^{1-\tau} \sim N^{-1} \]

\[ k_{\max} \sim N^{\frac{1}{\tau-1}} \]
The imposed cutoff can have a dramatic effect on the properties of the graph.
Generating functions for degree distributions

Random graphs with arbitrary degree distributions and their applications by Newman, Strogatz & Watts

\[ G_0(x) = \sum_{k=0}^{\infty} p_k x^k \] is a generating function

\[ p_k \sim k^{-\tau} \] is the probability that a randomly chosen vertex has degree \( k \)

\[ < k > = \sum_k k p_k = G'_0(1) \] is the expected degree of a randomly chosen vertex

\[ G_1(x) = \frac{G'_0(x)}{G'_0(1)} \] is the distribution of remaining outgoing edges following an edge

\[ z_2 = G'_0(1)G'_1(1) \] is the expected number of second degree neighbors assuming neighbors don’t share edges
search with knowledge of first neighbors

\[ G_0(x) = c \sum_{k=1}^{k_{\text{max}}} k^{-\tau} x^k \]  
Generating function with cutoff

\[ G'_0(x) = \frac{\partial}{\partial x} G_0(x) = c \sum_{k=1}^{k_{\text{max}}} k^{1-\tau} x^{k-1} \]  
Average degree of vertex

\[ G'_0(1) = \langle k \rangle = c \sum_{k=1}^{k_{\text{max}}} k^{1-\tau} : \int_1^{k_{\text{max}}} k^{1-\tau} dk = \frac{1}{\tau - 2} \left( 1 - k_{\text{max}}^{2-\tau} \right) \]

\[ G'_1(x) = \frac{G'_0(x)}{G'_0(1)} = \frac{c}{G'_0(1)} \frac{\partial}{\partial x} \sum_{k=1}^{k_{\text{max}}} k^{1-\tau} x^{k-1} \]  
Average number of neighbors following an edge

\[ = \frac{c}{G'_0(1)} \sum_{k=2}^{k_{\text{max}}} k^{1-\tau} (k - 1)x^{k-2} \]  
for \(2 < \tau < 3\), and \(k_{\text{max}} \sim N^\alpha\), decreases constant in \(N\) with \(N\)
In the limit $\tau > 2$, let's for the moment ignore the fact that as we do a random walk, we encounter neighbors that we’ve seen before.

$$s = \text{number of steps} = \frac{N}{z_{1B}}$$
Search time with different cutoffs

If \( k_{\text{max}} = N \),

\[
S(\tau) : \quad \frac{N}{k_{\text{max}}^{3-\tau}} = \frac{N}{N^{3-\tau}} = N^{\tau-2}, 2 < \tau < 3
\]

\( s(2.1) : N^{0.1} \)

\[
s : \quad \frac{N \log(k_{\text{max}})}{k_{\text{max}}} = \log(N), \tau = 2
\]

If \( k_{\text{max}} = N^{1/(\tau-1)} \),

\[
S(\tau) : \quad \frac{N}{k_{\text{max}}^{3-\tau}} = \frac{N}{N^{3-\tau}} = N^{2\frac{\tau-2}{\tau-1}}, 2 < \tau < 3
\]

\( s(2.1) : N^{0.18} \)

\[
s(2) : \quad \frac{N \log(k_{\text{max}})}{k_{\text{max}}} = \log(N)
\]
If $k_{\text{max}} = N^{1/\tau}$, $s : \frac{N}{k_{\text{max}}^{3-\tau}} = \frac{N}{(N^{1/\tau})^{3-\tau}} = N^{2-3/\tau}, 2 < \tau < 3$

So the best we can do is $\sqrt{N}$ for exponents close to 2

2nd neighbor random walk, ignoring overlap:

$$z_{2B} = \left[ \frac{\partial}{\partial x} G_i(G_i(x)) \right]_{x=1} = \left[ G'_i(1) \right]^2 = \left[ \frac{\tau - 2}{1 - k_{\text{max}}^{3-\tau}} \right]^2 (3 - \tau)$$

$$n_s = z_{2B}(N)$$

$$S \sim \frac{N}{z_{2B}(N)}$$

$$S(N, \tau) \sim N^{3(1-2/\tau)}$$

$$S(N, \tau = 2.1) \sim N^{0.15}$$
Following the degree sequence

Go to highest degree node, then next highest, ... etc.

\[ z_{1D} = \int_{k_{\text{max}}-a}^{k_{\text{max}}} N k^{1-\tau} \, dk \sim N a^{1-\tau} \]

\[ a \sim s = \# \text{ of steps taken} \]

2\textsuperscript{nd} neighbors, ignoring overlap:

\[ z_{1D} G_1(x) \sim N a^{2(2-\tau)} \]

\[ s \sim k_{\text{max}}^{2(\tau-2)} \sim N^{2-4/\tau} \]

\[ S_{\text{deg}} (N, \tau = 2.1) = N^{0.1} \]
Ratio of the degree of a node to the expected degree of its highest degree neighbor for 10,000 node power-law graphs of varying exponents.
Exponents $\tau$ close to 2 required to search effectively

**Gnutella**

**World Wide Web**, $\tau \sim 2.0$-$2.3$, high degree nodes: directories, search engines

**Social networks**, AT&T call graph $\tau \sim 2.1$

**Actor collaboration graph** (imdb database)

$\tau \sim 2.0$-$2.2$
Following the degree sequence
Complications

- Should not visit same node more than once

- Many neighbors of current node being visited were also neighbors of previously visited nodes, and there is a bias toward high degree nodes being ‘seen’ over and over again
Status and degree of node visited
Progress of exploration in a 10,000 node graph knowing 2nd degree neighbors

Seeking high degree nodes speeds up the search process.

About 50% of a 10,000 node graph is explored in the first 12 steps.
Scaling of search time with size of graph

- Random walk
- $\alpha = 0.37$ fit
- Degree sequence
- $\alpha = 0.24$ fit

Cover time for half the nodes vs. size of graph.
Comparison with a Poisson graph

$$G_0(x) = e^{x(x-1)}$$

$$G_1(x) = \frac{x}{z} G'_0(x) = G_0(x)$$

- Expected degree and expected degree following a link are equal.
- Scaling is linear.

constant av. deg. = 3.4

$\gamma = 1.0$ fit
Gnutella network

50% of the files in a 700 node network can be found in < 8 steps
Expander graphs

Time permitting
Def: Random k-Regular Graphs

We need to define two concepts

1) Define: Random k-Regular graph
   - Assume each node has $k$ spokes (half-edges)
   - Randomly pair them up!

2) Define: Expansion
   - Graph $G(V, E)$ has expansion $\alpha$:
     if $\forall S \subseteq V$: number of edges leaving $S$
     $\geq \alpha \cdot \min(|S|, |V \setminus S|)$
   - Or equivalently:
     $\alpha = \min_{S \subseteq V} \frac{\text{#edges leaving } S}{\min(|S|, |V \setminus S|)}$
Expansion: Intuition

\[ \alpha = \min_{S \subseteq V} \frac{\# \text{edges leaving } S}{\min(|S|, |V \setminus S|)} \]

(A big) graph with “good” expansion

Expansion: \( k \)-Regular Graphs

- **\( k \)-regular graph** (every node has degree \( k \)):  
  - Expansion is at most \( k \) (when \( S \) is a single node)

Is there a graph on \( n \) nodes (\( n \to \infty \)), of fixed max deg. \( k \), so that expansion \( \alpha \) remains const?

Examples:

- **\( n \times n \) grid**: \( k=4 \): \( \alpha = 2n/(n^2/4) \to 0 \)  
  (\( S = n/2 \times n/2 \) square in the center)

- **Complete binary tree**:  
  \( \alpha \to 0 \) for \( |S| = (n/2) - 1 \)

- **Fact**: For a random 3-regular graph on \( n \) nodes, there is some const \( \alpha \) (\( \alpha > 0 \), independent of \( n \)) such that w.h.p. the expansion of the graph is \( \geq \alpha \)  
  (In fact, \( \alpha = d/2 \) as \( d \to \infty \))

\[
\alpha = \min_{s \subseteq V} \frac{\text{#edges leaving } S}{\min(|S|, |V \setminus S|)}
\]
**Fact:** In a graph on $n$ nodes with expansion $\alpha$, for all pairs of nodes $s$ and $t$ there is a path of $O((\log n) / \alpha)$ edges connecting them.

**Proof:**

**Proof strategy:**

- We want to show that from any node $s$ there is a path of length $O((\log n)/\alpha)$ to any other node $t$.
- Let $S_j$ be a set of all nodes found within $j$ steps of BFS from $s$.
- How does $S_j$ increase as a function of $j$?
Proof (continued):

Let $S_j$ be a set of all nodes found within $j$ steps of BFS from $s$.

We want to relate $S_j$ and $S_{j+1}$

$$|S_{j+1}| \geq |S_j| + \frac{\alpha |S_j|}{k}$$

Expansion

At most $k$ edges "collide" at a node

$$|S_{j+1}| \geq |S_j| \left(1 + \frac{\alpha}{k}\right) = S_0 \left(1 + \frac{\alpha}{k}\right)^{j+1}$$

where $S_0 = 1$
Proof (continued):

In how many steps of BFS do we reach \(>n/2\) nodes?

Need \(j\) so that:

\[ S_j = \left( 1 + \frac{\alpha}{k} \right)^j \geq \frac{n}{2} \]

Let’s set:

\[ j = \frac{k \log_2 n}{\alpha} \]

Then:

\[ \left( 1 + \frac{\alpha}{k} \right)^{\alpha \frac{k \log_2 n}{\alpha}} \geq 2^{\log_2 n} = n > \frac{n}{2} \]

In \(2k/\alpha \cdot \log n\) steps \(|S_j|\) grows to \(\Theta(n)\).

So, the diameter of \(G\) is \(O(\log(n)/\alpha)\)

\[ e = \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x \]

Diameter = \(2^j\)

Make this into a 3-ary tree

Claim:

\[ \left( 1 + \frac{\alpha}{k} \right)^{\alpha \frac{k \log_2 n}{\alpha}} \geq 2^{\log_2 n} \]

Remember \(n>0, \alpha \leq k\) then:

if \(\alpha = k\):

\[ (1+1)^{\log_2 n} = 2^{\log_2 n} \]

if \(\alpha \to 0\) then

\[ \frac{k}{\alpha} \to \infty \]

and

\[ \left( 1 + \frac{1}{x} \right)^{x \log_2 n} = e^{\log_2 n} > 2^{\log_2 n} \]
Small world phenomenon:
- Local structure (e.g. clustering)
- Short average shortest path

The Watts-Strogatz captures both

Other models create *navigable* small-world models

Power-law networks are navigable due to presence of hubs