**Announcements:**

1) **HW3: Due 9:30am Friday 11/7**
   - Snap site was down (Gates network problems)
   - We give you 1 extra day if you are using a late period you still have to submit by Tue 9:30am)
2) HW4 has been posted
3) Project milestones due in 1 week
   - We expect ~50% project completed
   - Also no late periods

# Kronecker Graphs

CS224W: Social and Information Network Analysis
Jure Leskovec, Stanford University
http://cs224w.stanford.edu

# Macroscopic Evolution of Networks

# Macroscopic Evolution

- **How do networks evolve at the macro level?**
  - What are global phenomena of network growth?
- **Questions:**
  - What is the relation between the number of nodes $n(t)$ and number of edges $e(t)$ over time $t$?
  - How does diameter change as the network grows?
  - How does degree distribution evolve as the network grows?

# Network Evolution

- $\boldsymbol{N(t)}$ … nodes at time $\boldsymbol{t}$
- $\boldsymbol{E(t)}$ … edges at time $\boldsymbol{t}$
- Suppose that

$$\boldsymbol{N(t+1) = 2 \cdot N(t)}$$

- **Q:** what is:

$$\boldsymbol{E(t+1) = ?}$$ **Is it** $\boldsymbol{2 \cdot E(t)?}$

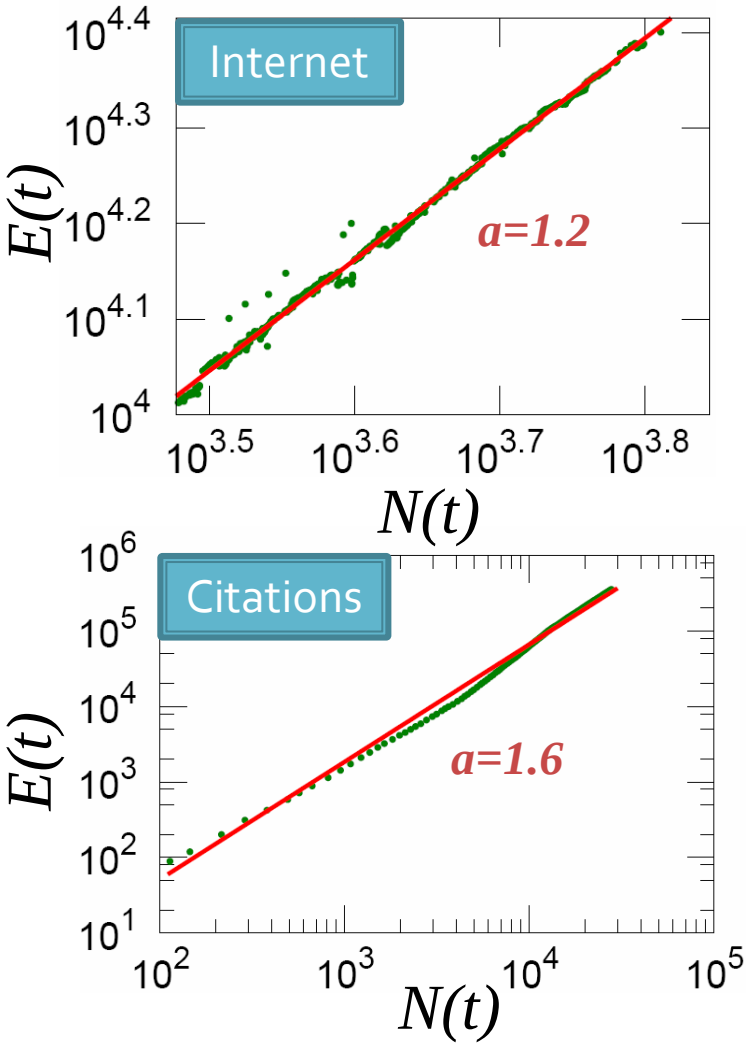
- **A: More than doubled!**
  - But obeying the **Densification Power Law**

# Q1) Network Evolution

- **What is the relation between the number of nodes and the edges over time?**
- ~~First guess: constant average degree over time~~
- Networks are **denser** over time
- **Densification Power Law:**

$$E(t) \propto N(t)^a$$

***a*** … densification exponent (1 ≤ **a** ≤ 2)



# Densification Power Law

- **Densification Power Law**
  - the number of edges grows faster than the number of nodes – **average degree is increasing**

$$E(t) \propto N(t)^a$$

or equivalently

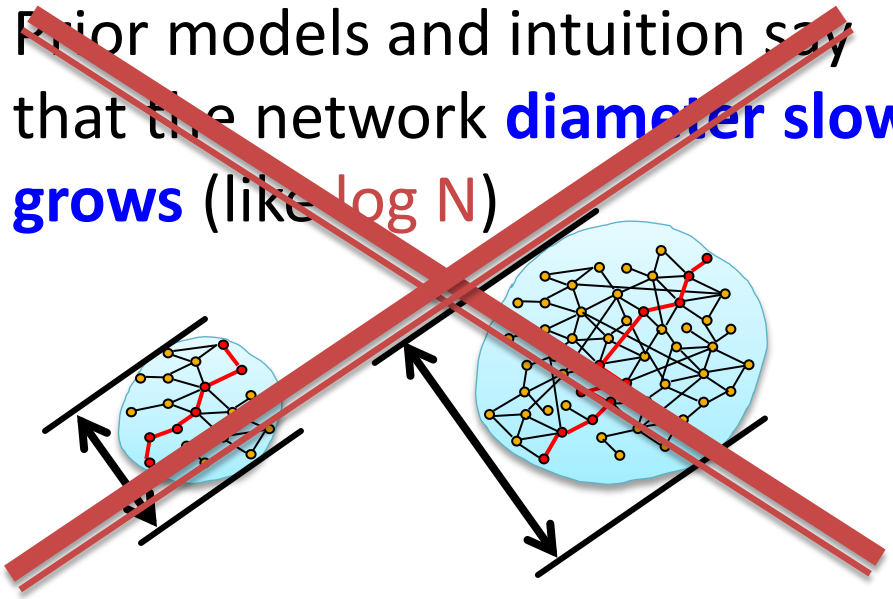
$$\frac{\log(E(t))}{\log(N(t))} = const$$

**a** … densification exponent: 1 ≤ **a** ≤ 2:

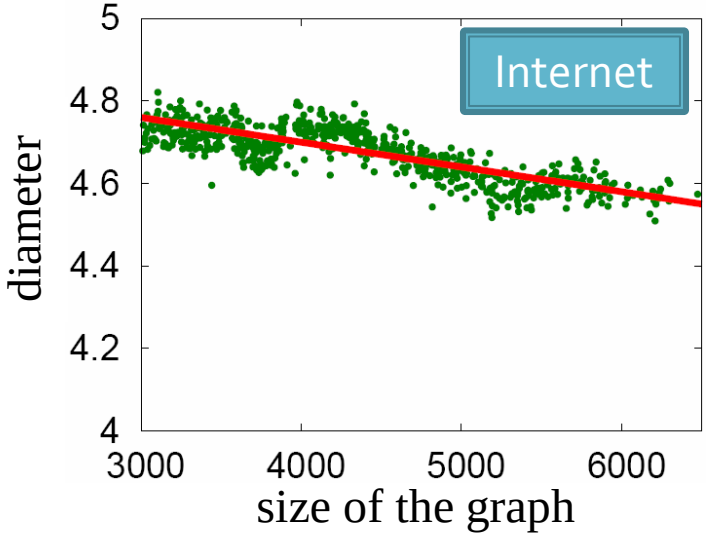
- **a=1: linear growth** – constant out-degree (traditionally assumed)
- **a=2: quadratic growth** – fully connected graph

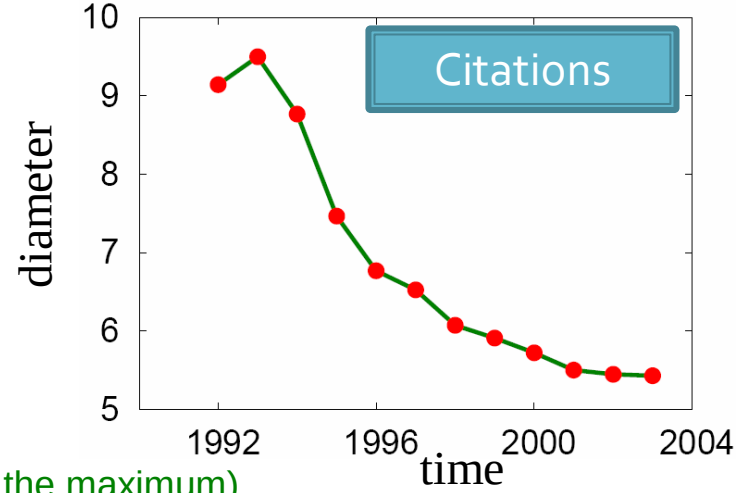
# Q1) Network Evolution

- ~~Prior models and intuition say that the network **diameter slowly grows** (like log N)~~



- **Diameter shrinks over time**
  - As the network grows the distances between the nodes slowly **decrease**



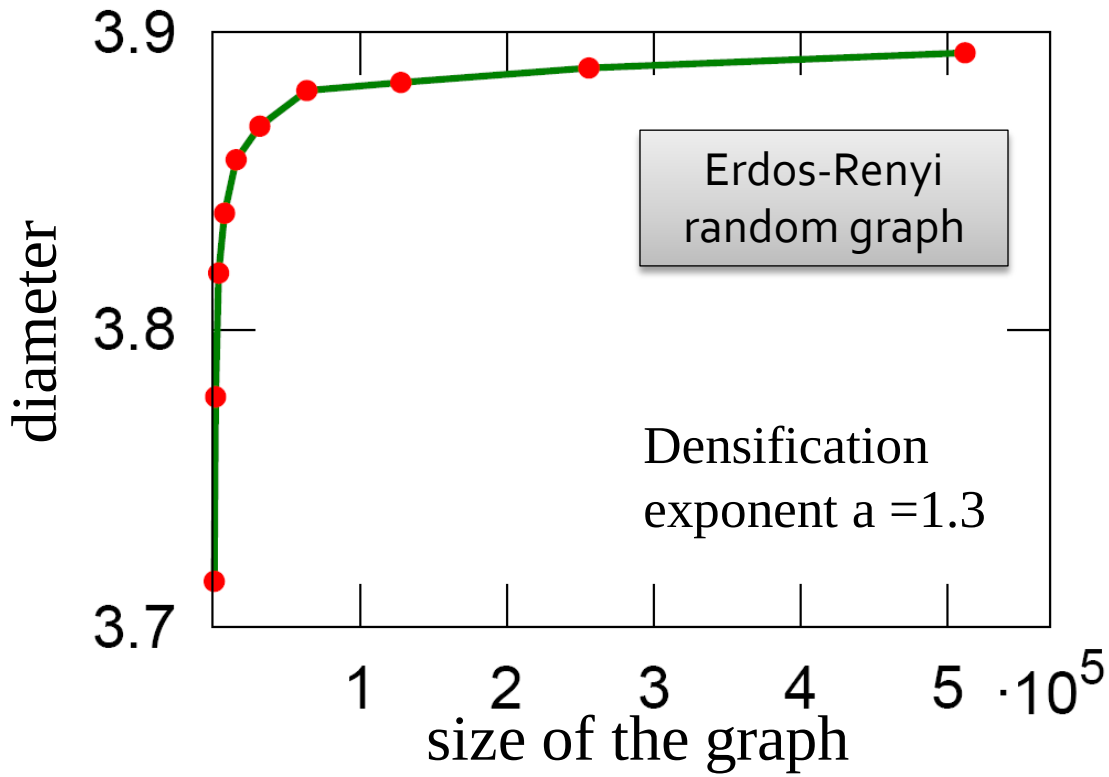

**How do we compute diameter in practice?**

**-- Long paths:** Take 90th-percentile or average path length (not the maximum)

**-- Disconnected components:** Take only largest component or average only over connected pairs of nodes

# Diameter of a Densifying $G_{np}$

Is shrinking diameter just a consequence of densification?



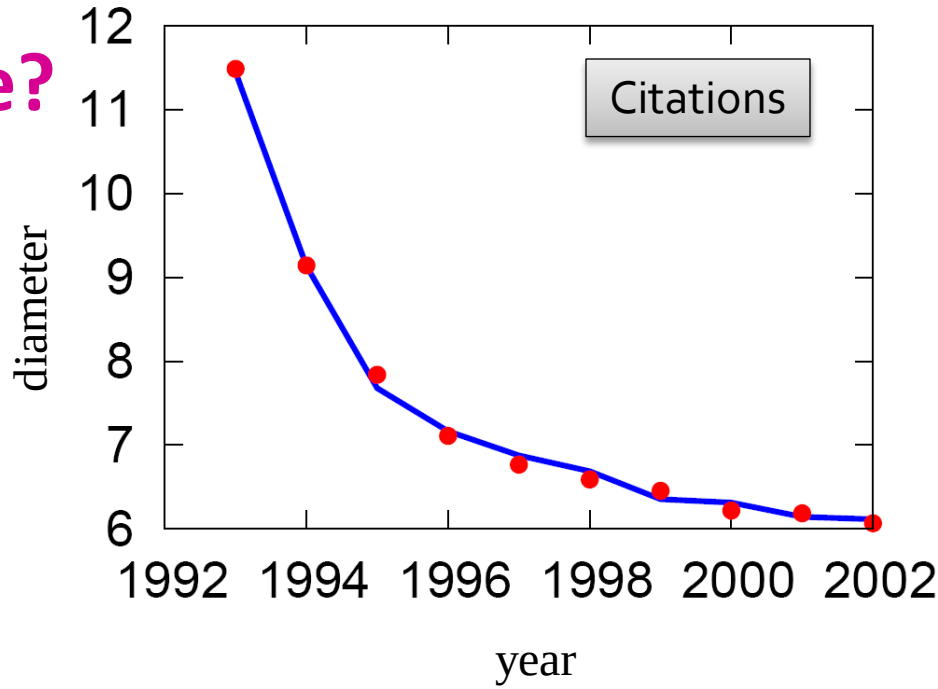

Densifying random graph has increasing diameter
$\Rightarrow$ **There is more to shrinking diameter than just densification!**

# Diameter of a Rewired Network

**Is it the degree sequence?**

Compare diameter of a:

- Real network (**red**)
- Random network with the same degree distribution (**blue**)

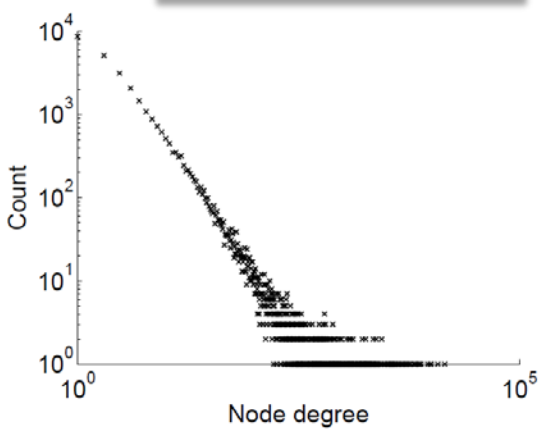


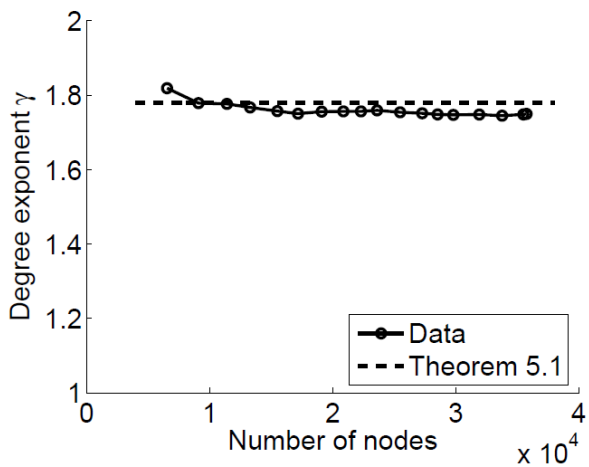

**Densification + degree sequence gives shrinking diameter**

# Connecting Degrees & Densification

- **How does degree distribution evolve to allow for densification?**
- **Option 1) Degree exponent $\gamma_t$ is constant:**
  - **Fact 1:** If $\gamma_t = \gamma \in [1, 2]$, then: $\alpha = 2/\gamma$

Email network



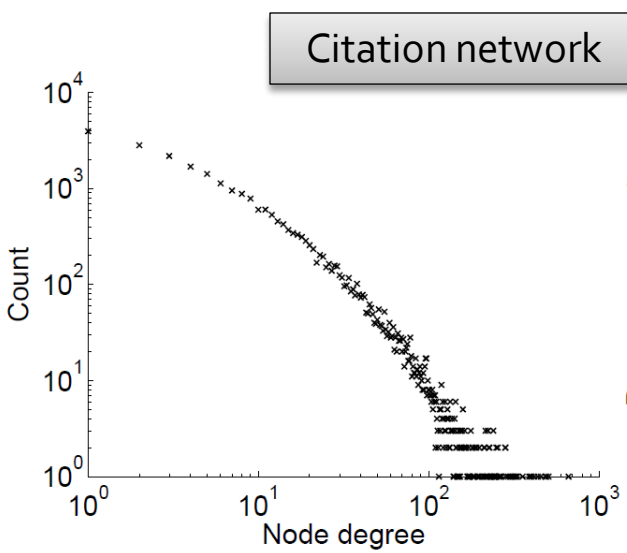

**A consequence of what we learned in the Power law lecture**:

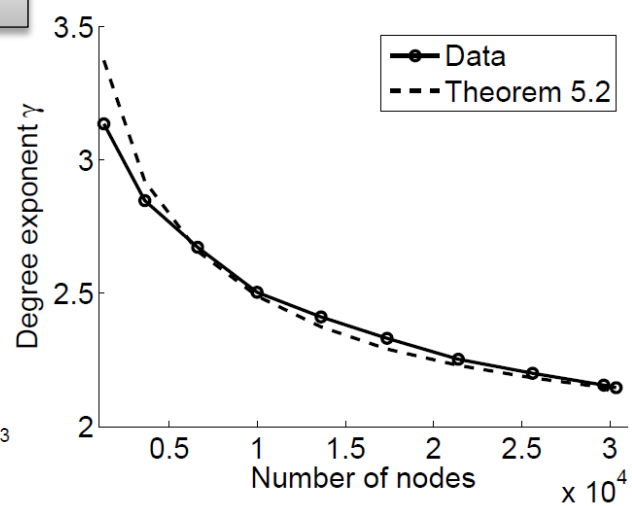
- Power-laws with exponents <2 have infinite expectations.
- So, by maintaining constant degree exponent $\alpha$ the average degree grows.

# Connecting Degrees & Densification

- **How does degree distribution evolve to allow for densification?**
- **Option 2) $\gamma_t$ evolves with graph size $n$:**
  - **Fact 2:** If $\gamma_t = \frac{4n_t^{x-1}-1}{2n_t^{x-1}-1}$, then: $a = x$

Notice: $\gamma_t \to 2$ as $n_t \to \infty$




Remember, the expected degree in a power law is:

$$E[X] = \frac{\gamma_t - 1}{\gamma_t - 2} x_m$$

So $\gamma_t$ has to decay as a function of graph size $n_t$ for the avg. degree to go up.

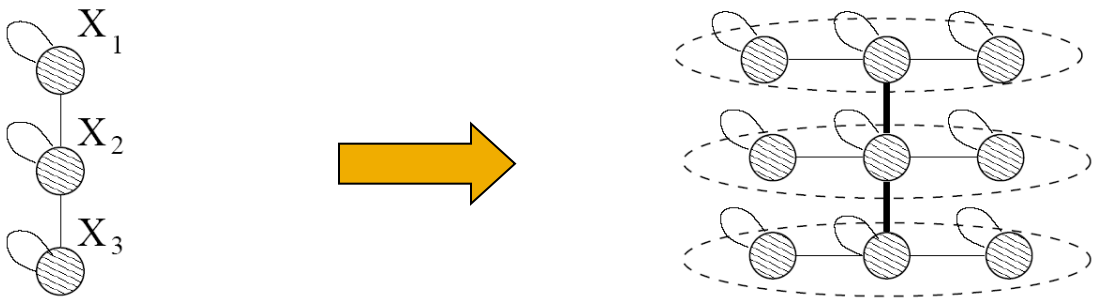
# Kronecker Graphs Model

# Models of Networks

- **What is the goal of modeling networks?**
  - **Discover structural properties of networks**
    - Small-world, Edge clustering, Heavy-tailed degrees
  - **Find a model that gives graphs with such properties**
    - Erdos-Renyi, Watts-Strogatz, Barabasi-Albert model
- **Today's lecture:**
  - **Can we have a model that attempts to reproduce all of these properties?**
  - **Can we fit the model to a network and accurately reproduce the network?**

# Idea: Recursive Graph Generation

- **How can we think of network structure recursively? Intuition:** Self-similarity
  - **Object is similar to a part of itself**: the whole has the same shape as one or more of the parts
- **Mimic recursive graph/community growth:**

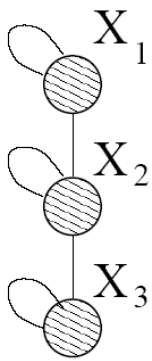


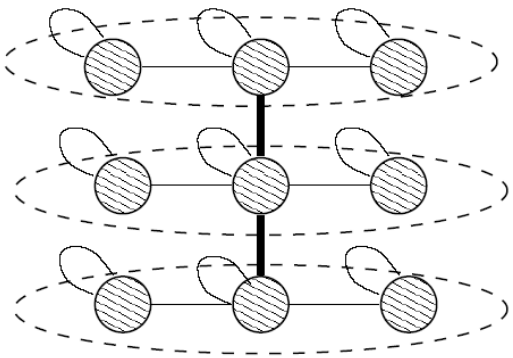

Initial graph  Recursive expansion

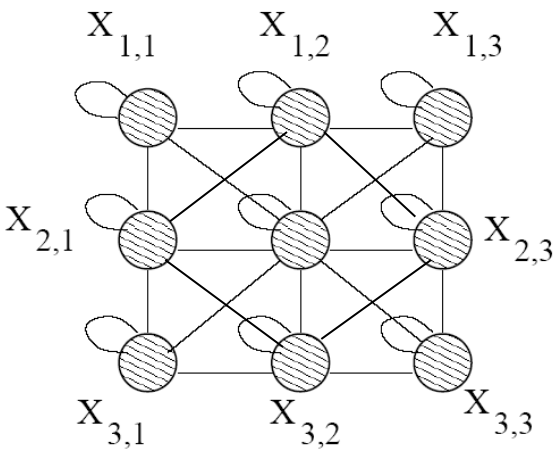
- **Kronecker graph** is a way of generating self-similar matrices

[PKDD '05]

# Kronecker: Graph Growth






Intermediate stage

| | | |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 1 | 1 |
| 0 | 1 | 1 |

(3x3)

$K_1$

Initiator graph

| | | |
|---|---|---|
| $K_1$ | $K_1$ | 0 |
| $K_1$ | $K_1$ | $K_1$ |
| 0 | $K_1$ | $K_1$ |

(9x9)

$K_2 = K_1 \otimes K_1$

After the growth phase

[PKDD '05]

# Kronecker Graph

- **Kronecker graphs:**
  - **A recursive model of network structure**

| | | |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 1 | 1 |
| 0 | 1 | 1 |

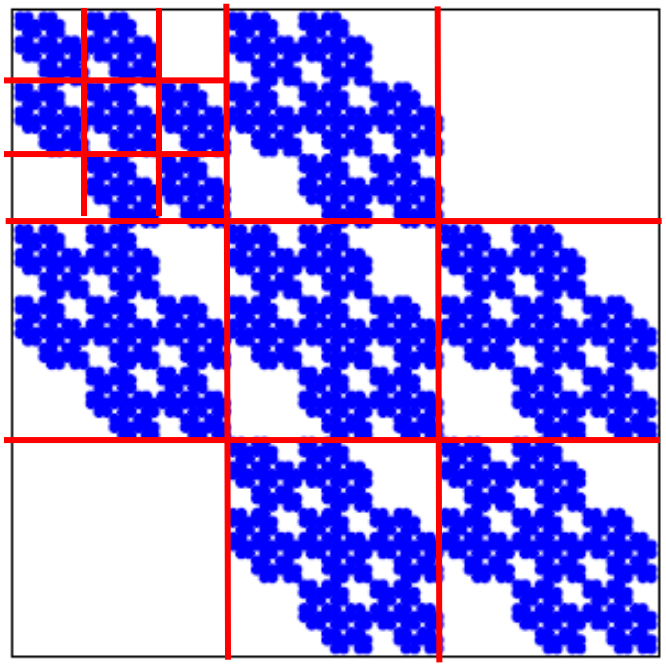
$K_1$

3 x 3

| | | |
|---|---|---|
| $K_1$ | $K_1$ | 0 |
| $K_1$ | $K_1$ | $K_1$ |
| 0 | $K_1$ | $K_1$ |

$K_2 = K_1 \otimes K_1$

9 x 9



81 x 81 adjacency matrix

# Kronecker Product: Definition

- **Kronecker product** of matrices $A$ and $B$ is given by

$$\underset{N \times M}{\mathbf{C}} = \underset{K \times L}{\mathbf{A} \otimes \mathbf{B}} \doteq \underset{N*K \times M*L}{\begin{pmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} & \dots & a_{1,m}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} & \dots & a_{2,m}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}\mathbf{B} & a_{n,2}\mathbf{B} & \dots & a_{n,m}\mathbf{B} \end{pmatrix}}$$

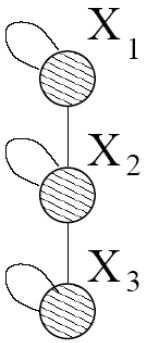
- Define a Kronecker product of two **graphs** as a Kronecker product of their **adjacency matrices**

[PKDD '05]

# Kronecker Graphs

- **Kronecker graph:** a growing sequence of graphs by iterating the **Kronecker product:**

| | | |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 1 | 1 |
| 0 | 1 | 1 |



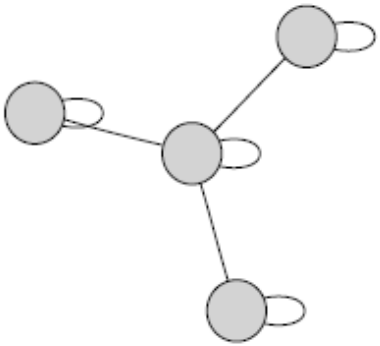

$K_1$

$$K_1^{[m]} = K_m = \underbrace{K_1 \otimes K_1 \otimes \ldots K_1}_{m \ \textit{times}} = K_{m-1} \otimes K_1$$

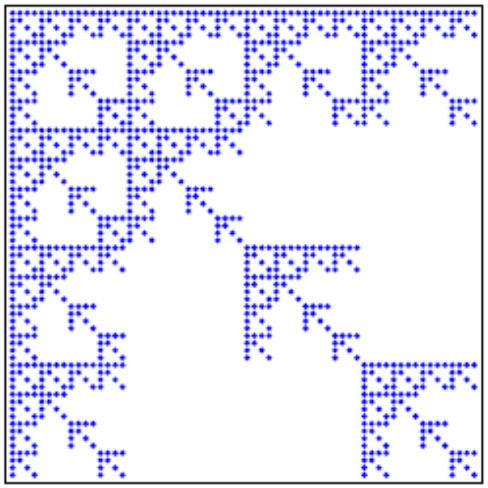
- **Note:** One can easily use multiple initiator matrices $(K_1', K_1'', K_1''')$ (even of different sizes)

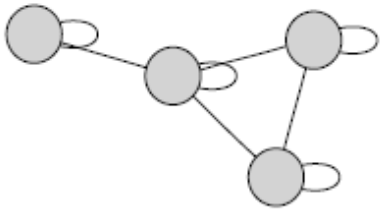
[PKDD '05]

# Kronecker Initiator Matrices

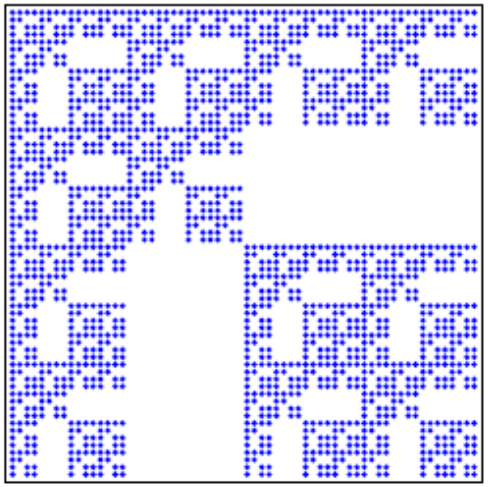


| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 |





| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |



Initiator $K_1$ | $K_1$ adjacency matrix | $K_3$ adjacency matrix

[PKDD '05]

# Kronecker Graphs: First Fun Fact

$$K_1^{[m]} = K_m = \underbrace{K_1 \otimes K_1 \otimes \ldots K_1}_{m \ times} = K_{m-1} \otimes K_1$$

**First fact about Kronecker Graphs!**

- For $\boldsymbol{K_1}$ on $\boldsymbol{N_1}$ nodes and $\boldsymbol{E_1}$ edges $\boldsymbol{K_m}$ ($m^{\text{th}}$ Kronecker power of $\boldsymbol{K_1}$) has:
  - $\boldsymbol{N(m) = N_1{}^m}$ nodes
  - $\boldsymbol{E(m) = E_1{}^m}$ edges
- So, we get the **densification power-law!**
  - $\boldsymbol{E(t) \propto N(t)^a}$, **so:** $E_1^t = (N_1^t)^a$ What is $\boldsymbol{a}$?
  - $\boldsymbol{a = \frac{\log(E(t))}{\log(N(t))} = \frac{log(E_1^t)}{log(N_1^t)} = \frac{\log(E_1)}{\log(N_1)}}$

| | | |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 1 | 1 |
| 0 | 1 | 1 |

$K_1$

Since E(t) > N(t), then $\boldsymbol{a > 1}$

[PKDD '05]

# Properties of Kronecker Graphs

- **Properties of deterministic Kronecker graphs (can be proved!)**
  - **Properties of static networks:**
    - Power-Law like Degree Distribution
    - Power-Law eigenvalue and eigenvector distribution
    - Constant Diameter
  - **Properties of evolving networks:**
    - Densification Power Law **(just proved)**
    - Shrinking/Stabilizing Diameter (for Stochastic Kronecker graphs)

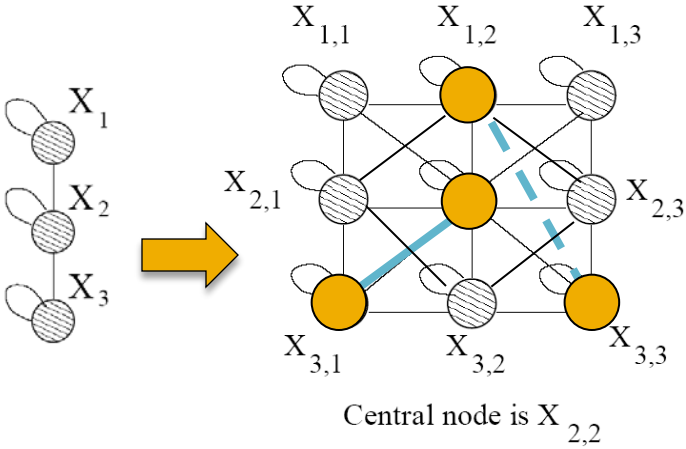
[PKDD '05]

# Constant Diameter

- **Observation:** Edges in Kronecker graphs:

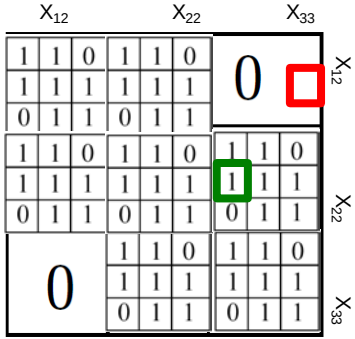
$$Edge\ (X_{ij}, X_{kl}) \in G \otimes H$$
$$iff\ (X_i, X_k) \in G\ and\ (X_j, X_l) \in H$$

where $\boldsymbol{X}$ are appropriate nodes in $\boldsymbol{G}$ and $\boldsymbol{H}$




- **Why?**
  - An entry in matrix **G⊗H** is a multiplication of entries in $\boldsymbol{G}$ and $\boldsymbol{H}$.

| | | |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 1 | 1 |
| 0 | 1 | 1 |



[PKDD '05]

# Constant Diameter

- **<u>Theorem:</u> Constant diameter**: If graphs $G, H$ have diameter $d$ then $G \otimes H$ has diameter $d$
- **What is distance between nodes $u, v$ in $G \otimes H$?**
  - Consider some nodes $u = [a, b], v = [a', b']$ in $G \otimes H$
  - Then, path $a$ to $a'$ in $G$ is less $d$ steps: $\boxed{a_1, a_2}, a_3, \ldots, a_d$
  - And path $b$ to $b'$ in $H$ is less $d$ steps: $\boxed{b_1, b_2}, b_3, \ldots, b_d$
  - **How many steps from u to v?**
    - **We know** edge $([a_1, b_1], [a_2, b_2])$ is in $G \otimes H$
    - So it takes $< d$ steps to get from u to v in $G \otimes H$
- **Consequence:**
  - If $K_1$ has diameter $d$ then graph $K_k$ also has diameter $d$

# Stochastic Kronecker Graphs

[PKDD '05]

# Stochastic Kronecker Graphs

- Create $N_1 \times N_1$ **probability matrix** $\boldsymbol{\Theta_1}$
- Compute the $k^{th}$ Kronecker power $\boldsymbol{\Theta_k}$
- For each entry $p_{uv}$ of $\boldsymbol{\Theta_k}$ include an edge $(u, v)$ in $K_k$ with probability $p_{uv}$

| | |
|---|---|
| 0.5 | 0.2 |
| 0.1 | 0.3 |

$\Theta_1$

Kronecker multiplication

| | | | |
|---|---|---|---|
| 0.25 | 0.10 | 0.10 | 0.04 |
| 0.05 | 0.15 | 0.02 | 0.06 |
| 0.05 | 0.02 | 0.15 | 0.06 |
| 0.01 | 0.03 | 0.03 | 0.09 |

$\Theta_2 = \Theta_1 \otimes \Theta_1$

Probability of edge $p_{uv}$

**Instance matrix $K_2$**

Flip biased coins

# Generation of Kronecker Graphs

- **How do we generate an instance of a stochastic Kronecker graph?**

Probability of edge $p_{uv}$

| | | | |
|---|---|---|---|
| 0.25 | 0.10 | 0.10 | 0.04 |
| 0.05 | 0.15 | 0.02 | 0.06 |
| 0.05 | 0.02 | 0.15 | 0.06 |
| 0.01 | 0.03 | 0.03 | 0.09 |

Flip biased coins

| | | | |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |

Need to flip $n^2$ coins!! Way too slow!!

- **Is there a faster way? YES!**
- **Idea:** Exploit the recursive structure of Kronecker graphs
  - "Drop" edges one by one

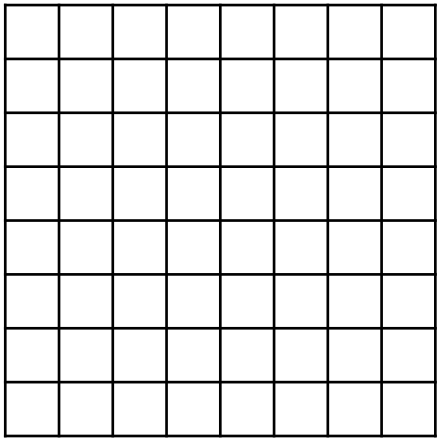
# Generation of Kronecker Graphs

- **A faster way to generate Kronecker graphs**

$$\Theta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & a{\cdot}a & a{\cdot}b & b{\cdot}a & b{\cdot}b \\ v_2 & a{\cdot}c & a{\cdot}d & b{\cdot}c & b{\cdot}d \\ v_3 & c{\cdot}a & c{\cdot}b & d{\cdot}a & d{\cdot}b \\ v_4 & c{\cdot}c & c{\cdot}d & d{\cdot}c & d{\cdot}d \end{array} = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & a & b & a & b \\ v_2 & c & d & c & d \\ v_3 & a & b & a & b \\ v_4 & c & d & c & d \end{array}$$

$\Theta \otimes \Theta$

- **How to "drop" an edge into a graph $G$ on $n = 2^m$ nodes**



Adjacency matrix **G**

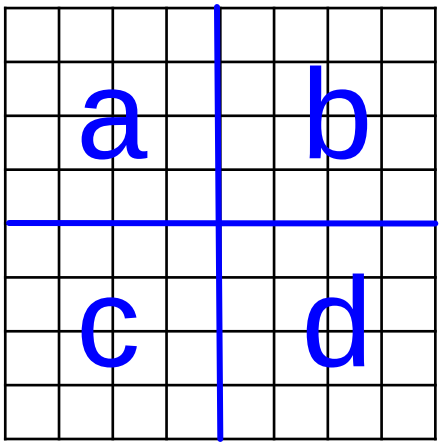
# Generation of Kronecker Graphs

- **A faster way to generate Kronecker graphs**

$$\Theta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & a{\cdot}a & a{\cdot}b & b{\cdot}a & b{\cdot}b \\ v_2 & a{\cdot}c & a{\cdot}d & b{\cdot}c & b{\cdot}d \\ v_3 & c{\cdot}a & c{\cdot}b & d{\cdot}a & d{\cdot}b \\ v_4 & c{\cdot}c & c{\cdot}d & d{\cdot}c & d{\cdot}d \end{array} = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & a & b & a & b \\ v_2 & c & d & c & d \\ v_3 & a & b & a & b \\ v_4 & c & d & c & d \end{array}$$

$$\Theta \otimes \Theta$$

- **How to "drop" an edge into a graph $G$ on $n = 2^m$ nodes**



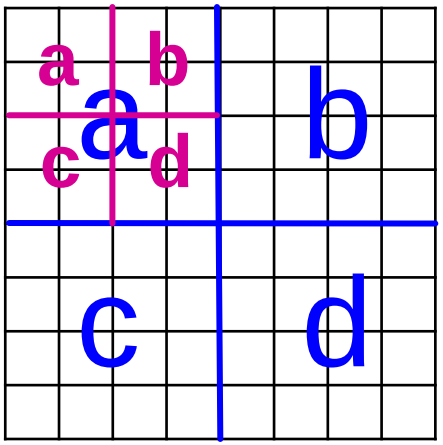

Adjacency matrix **G**

# Generation of Kronecker Graphs

- **A faster way to generate Kronecker graphs**

$$\Theta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \underbrace{\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & a{\cdot}a & a{\cdot}b & b{\cdot}a & b{\cdot}b \\ v_2 & a{\cdot}c & a{\cdot}d & b{\cdot}c & b{\cdot}d \\ v_3 & c{\cdot}a & c{\cdot}b & d{\cdot}a & d{\cdot}b \\ v_4 & c{\cdot}c & c{\cdot}d & d{\cdot}c & d{\cdot}d \end{array}}_{\Theta \otimes \Theta} = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & a & b & a & b \\ v_2 & c & d & c & d \\ v_3 & a & b & a & b \\ v_4 & c & d & c & d \end{array}$$

- **How to "drop" an edge into a graph $G$ on $n = 2^m$ nodes**



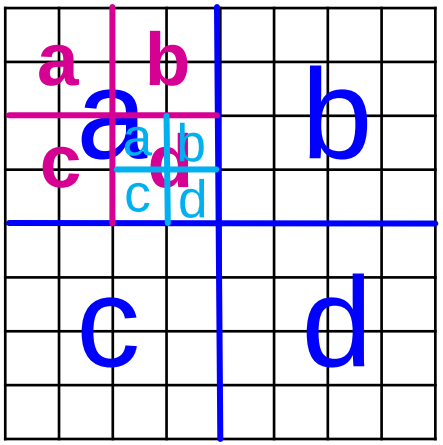

Adjacency matrix **G**

# Generation of Kronecker Graphs

- **A faster way to generate Kronecker graphs**

$$\Theta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \underbrace{\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & a{\cdot}a & a{\cdot}b & b{\cdot}a & b{\cdot}b \\ v_2 & a{\cdot}c & a{\cdot}d & b{\cdot}c & b{\cdot}d \\ v_3 & c{\cdot}a & c{\cdot}b & d{\cdot}a & d{\cdot}b \\ v_4 & c{\cdot}c & c{\cdot}d & d{\cdot}c & d{\cdot}d \end{array}}_{\Theta \otimes \Theta} = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & a & b & a & b \\ v_2 & c & d & c & d \\ v_3 & a & b & a & b \\ v_4 & c & d & c & d \end{array}$$

- **How to "drop" an edge into a graph $G$ on $n = 2^m$ nodes:**
  - We may get a few edges colliding. We simply reinsert them.

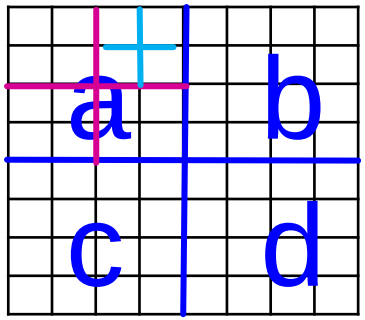



Adjacency matrix **G**

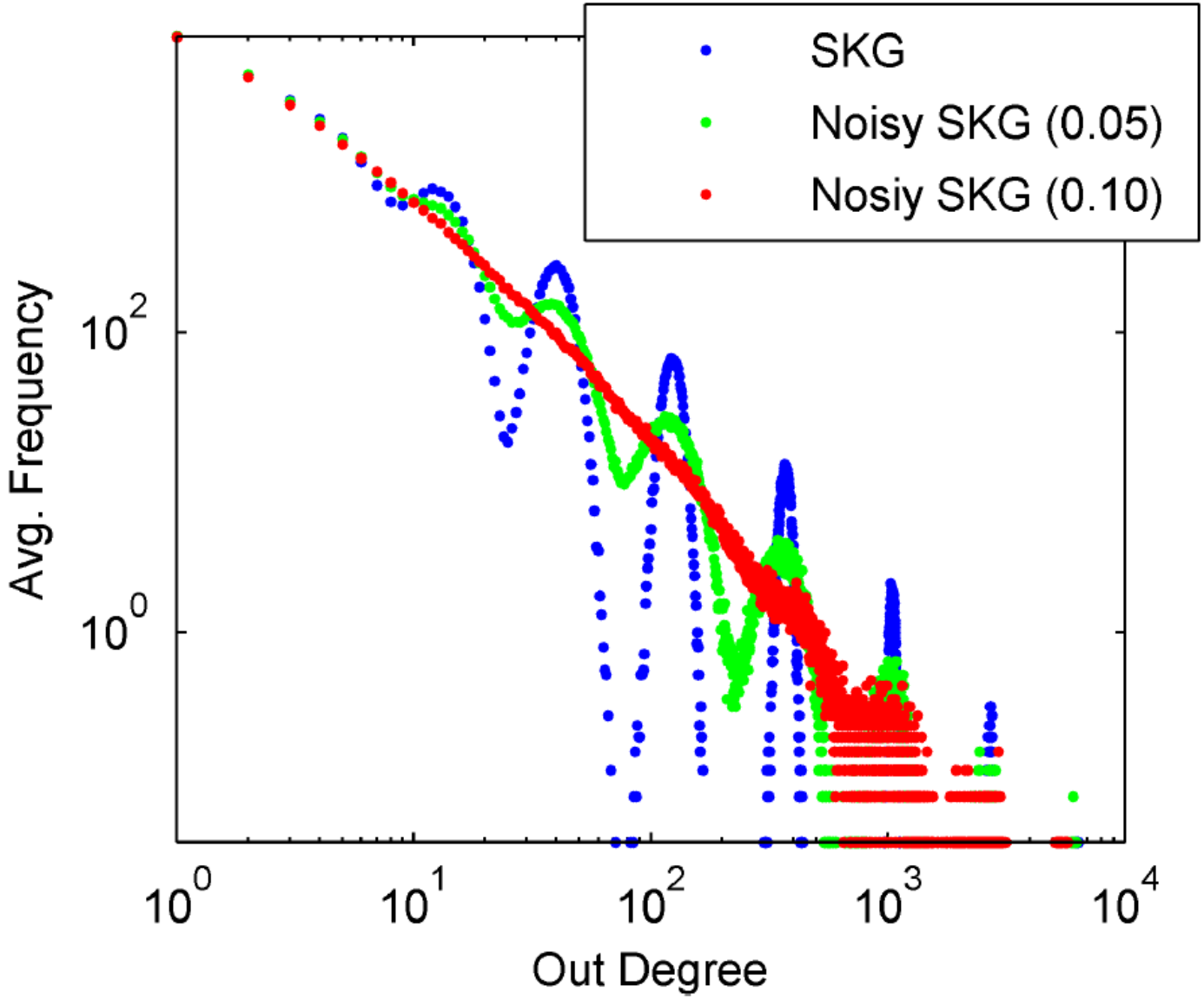
# Generation of Kronecker Graphs

## Fast Kronecker generator algorithm:

- **Insert 1 edge on graph $G$ on $n = 2^m$ nodes:**
  - Create normalized matrix $L_{uv} = \Theta_{uv}/(\sum_{op} \Theta_{op})$
  - **For $i = 1 \ldots m$**
    - start with $x = 0, y = 0$
    - Pick an row/column $(u, v)$ with prob. $L_{uv}$
    - Descend into quadrant $(u, v)$ at level $i$ of $G$
      - **This means:** $x \mathrel{+}= u \cdot 2^{m-i}$ , $y \mathrel{+}= v \cdot 2^{m-i}$
  - Add an edge $G[x, y] = 1$



# Problem: Spikes in Node Degrees!



[Seshadhri et al. '13]

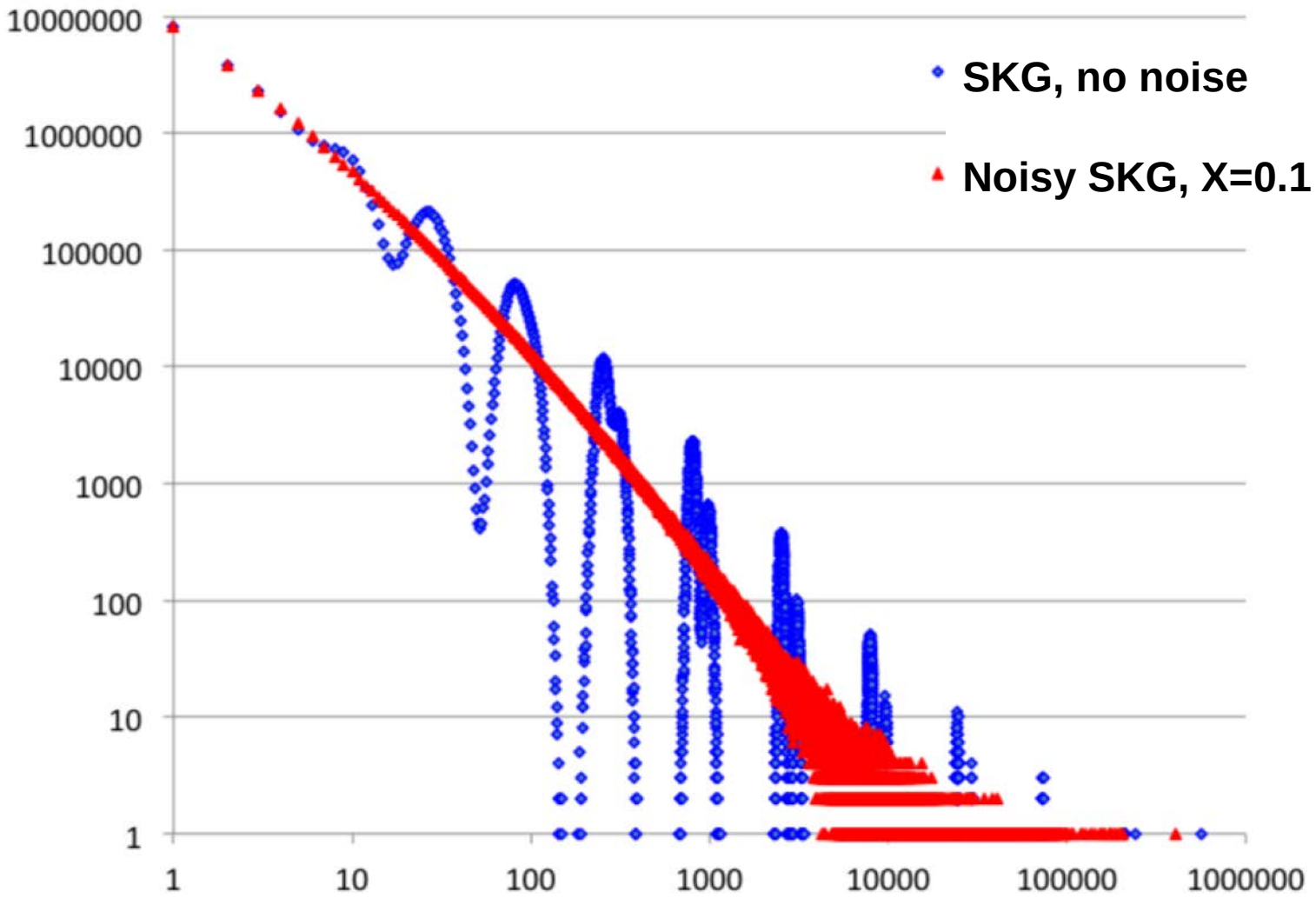
# Solution: Noisy SKG

- **Solution: Noisy Stochastic Kronecker Graphs**
  - **Idea:** Add noise to the matrix $\Theta$
    - There are many ways how one could do this, but here is the correct way!
  - Assume $\boldsymbol{\Theta} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and **G** has $\mathbf{2^m}$ nodes
  - **Then create "noisy" matrices** $\Theta_1 \ldots \Theta_m$ where:
  - $\Theta_i = \begin{bmatrix} a - \frac{2x_i a}{a+d} & b + x_i \\ c + x_i & d - \frac{2x_i d}{a+d} \end{bmatrix}$

    **Where $x_i$ is a random number on interval [-X, +X]**
    **And X is the noise level.**
  - Apply Kronecker generator to this set of matrices

[Seshadhri et al. '13]

# SKG Degree Distribution



[Mahdian-Xu, WAW '07]

# Stochastic Kronecker Graphs

## What is known about Stochastic Kronecker?

- **Undirected** Kronecker graph model with:
  - **Connected**, if:
    - $b + c > 1$
  - **Connected component of size $\Theta(n)$**, if:
    - $(a + b)(b + c) > 1$
  - **Constant diameter**, if:
    - $b + c > 1$
  - **Not searchable** by a decentralized algorithm

$$\Theta_1 = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$a > b > c$

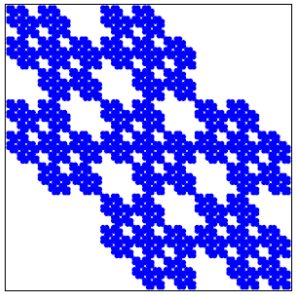
[ICML '07]

# Kronecker Graphs: Estimation

$$\Theta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

## How to estimate $\Theta$ given a $G$?

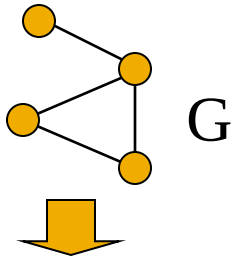
- **KronFit: Maximum likelihood estimation**
- Given real graph $G$
- Find Stochastic Kronecker initiator $\Theta$ which

$\arg\max_{\Theta} P($  $|$  $\xleftarrow{\text{Kronecker}}$ $\Theta)$

- **To solve this we need to:**
  - Efficiently calculate $\boldsymbol{P(G|\Theta)}$
  - Then maximize over $\boldsymbol{\Theta}$ (e.g., using gradient descent)

# KronFit: Likelihood $P(G|\Theta)$

- Given $G$ and $\Theta$ we calculate likelihood that $\Theta$ generated $G$: $\boldsymbol{P(G|\Theta)}$




| | |
|---|---|
| 0.5 | 0.2 |
| 0.1 | 0.3 |

$\Theta$

| | | | |
|---|---|---|---|
| 0.25 | 0.10 | 0.10 | 0.04 |
| 0.05 | 0.15 | 0.02 | 0.06 |
| 0.05 | 0.02 | 0.15 | 0.06 |
| 0.01 | 0.03 | 0.03 | 0.09 |

$\Theta_k$

| | | | |
|---|---|---|---|
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 |

$G$

$\boldsymbol{P(G|\Theta)}$

$$P(G\,|\,\Theta) = \prod_{(u,v)\in G} \Theta_k[u,v] \prod_{(u,v)\notin G} (1-\Theta_k[u,v])$$

Likelihood of edges in the graph  Likelihood of edges not in the graph

# Challenge 1: Node Correspondence

$\Theta$

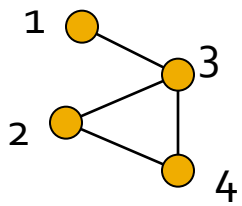
| | |
|---|---|
| 0.5 | 0.2 |
| 0.1 | 0.3 |

$\Theta_k$

| | | | |
|---|---|---|---|
| 0.25 | 0.10 | 0.10 | 0.04 |
| 0.05 | 0.15 | 0.02 | 0.06 |
| 0.05 | 0.02 | 0.15 | 0.06 |
| 0.01 | 0.03 | 0.03 | 0.09 |

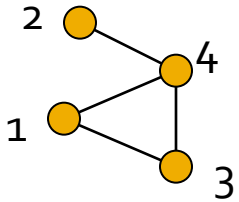
$\sigma$

$G'$




| | | | |
|---|---|---|---|
| 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 |

$G''$




| | | | |
|---|---|---|---|
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 |

$$P(G'|\Theta) = P(G''|\Theta)$$

- Nodes are **unlabeled**
- Graphs ***G'*** and ***G''*** should have the same likelihood $\boldsymbol{P(G'|\Theta) = P(G''|\Theta)}$
- One needs to consider all node correspondences $\boldsymbol{\sigma}$

$$P(G\,|\,\Theta) = \sum_{\sigma} P(G\,|\,\Theta,\sigma)P(\sigma)$$

- All correspondences are a priori equally likely
- There are ***O(n!)*** correspondences

[ICML '07]

# Challenge 2: Calculating $P(G|\Theta,\sigma)$

- **Assume that we solved the node correspondence problem**
- **Calculating:**

$$P(G\,|\,\Theta) = \prod_{(u,v)\in G} \Theta_k[u,v] \prod_{(u,v)\notin G} (1-\Theta_k[u,v])$$

- Takes ***O(n²)*** time!

| | | | |
|---|---|---|---|
| 0.25 | 0.10 | 0.10 | 0.04 |
| 0.05 | 0.15 | 0.02 | 0.06 |
| 0.05 | 0.02 | 0.15 | 0.06 |
| 0.01 | 0.03 | 0.03 | 0.09 |

$\Theta_k$

$\sigma$

| | | | |
|---|---|---|---|
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 |

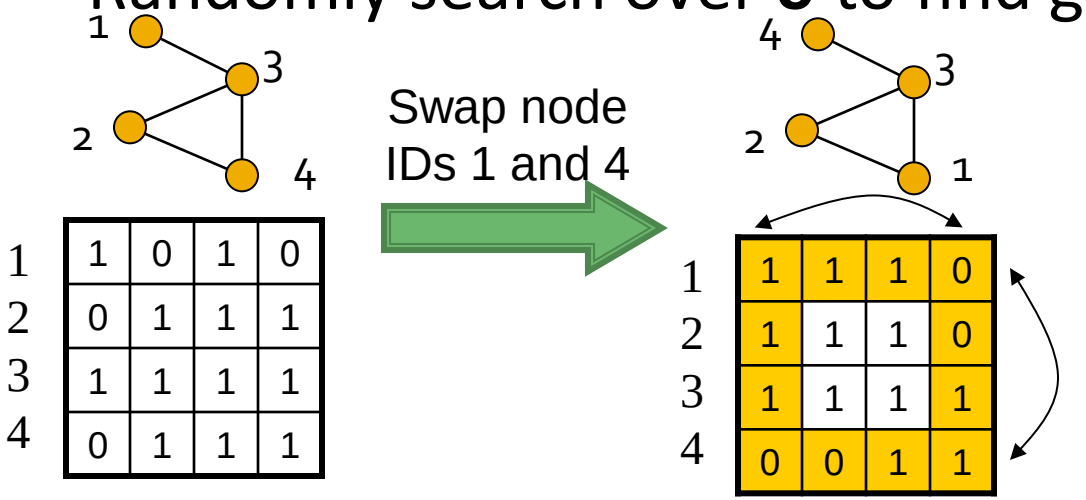
$G$

$$\boxed{P(G|\Theta,\,\sigma)}$$

# KronFit: Solution Overview

Details in Leskovec-Faloutsos, ICML '07

- **Node correspondence:**
  - Node permutation **σ** defines the mapping
  - Randomly search over **σ** to find good mappings




**The algorithm (called Metropolis sampling):**
**(1)** Pick 2 nodes at random
**(2)** Swap their IDs
**(3)** Does it improve the fit $P(G|\Theta,\sigma)$? **If, yes, keep the swap, else undo it**
**(4)** Go to (1)

- **Calculating the likelihood $P(G|\Theta,\sigma)$**
  - Calculate likelihood of **empty graph** (**G** with **0** edges)
  - Correct it for edges that we observe in the graph

# Experiments: real networks

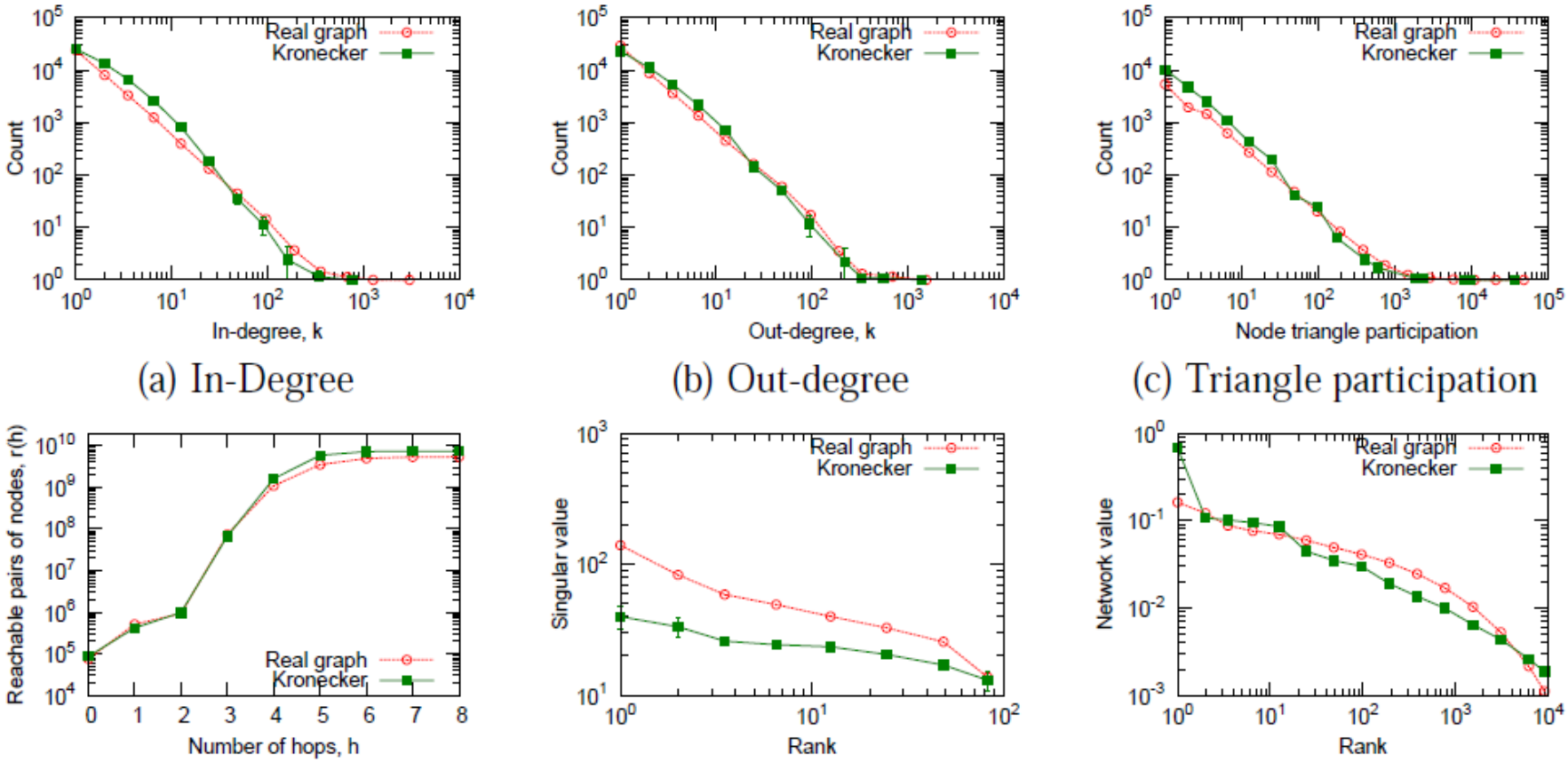
$$\Theta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- **Experimental setup**
  - Given real graph $\boldsymbol{G}$
  - Estimate parameters $\boldsymbol{\Theta}$
  - Generate synthetic graph $\boldsymbol{K}$ using $\boldsymbol{\Theta}$
  - Compare properties of graphs $\boldsymbol{G}$ and $\boldsymbol{K}$
- **Note:**
  - We do not fit the graph properties themselves
  - We fit the likelihood and then compare the properties

# Estimation: Epinions (n=76k, m=510k)

- **Real** and **Kronecker** are very close:

$$\Theta_1 = \begin{bmatrix} 0.99 & 0.54 \\ 0.49 & 0.13 \end{bmatrix}$$




(a) In-Degree (b) Out-degree (c) Triangle participation

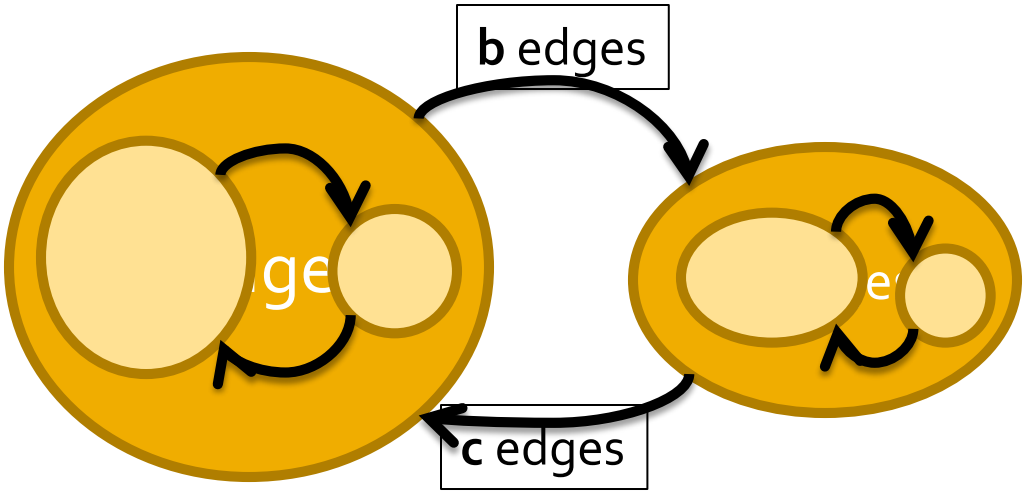
(d) Hop plot (e) Scree plot (f) "Network" value

[JMLR '10]

# Kronecker & Network Structure

- **What do estimated parameters tell us about the network structure?**

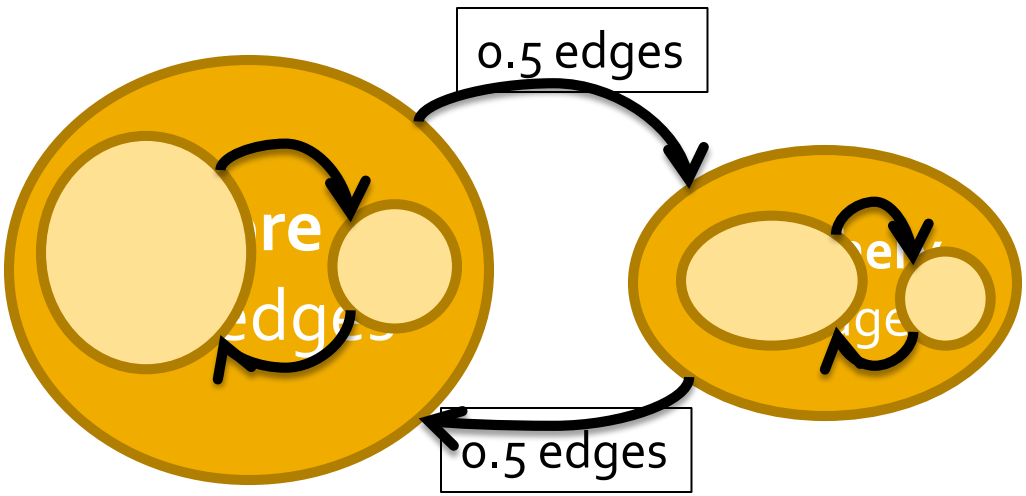
$$\Theta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

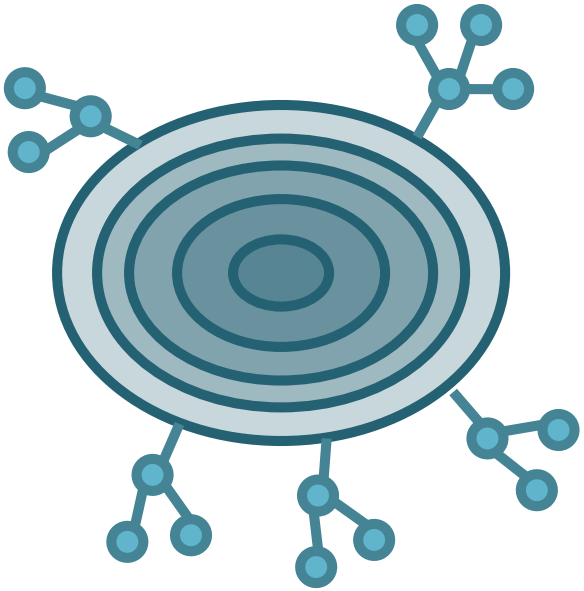


# Kronecker & Network structure

- **What do estimated parameters tell us about the network structure?**

$$\Theta = \begin{bmatrix} 0.9 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}$$



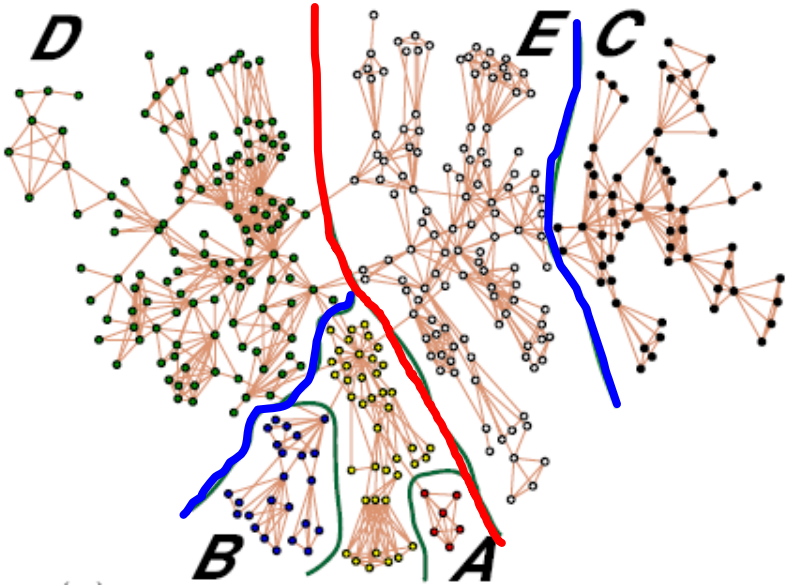



**Nested Core-periphery**

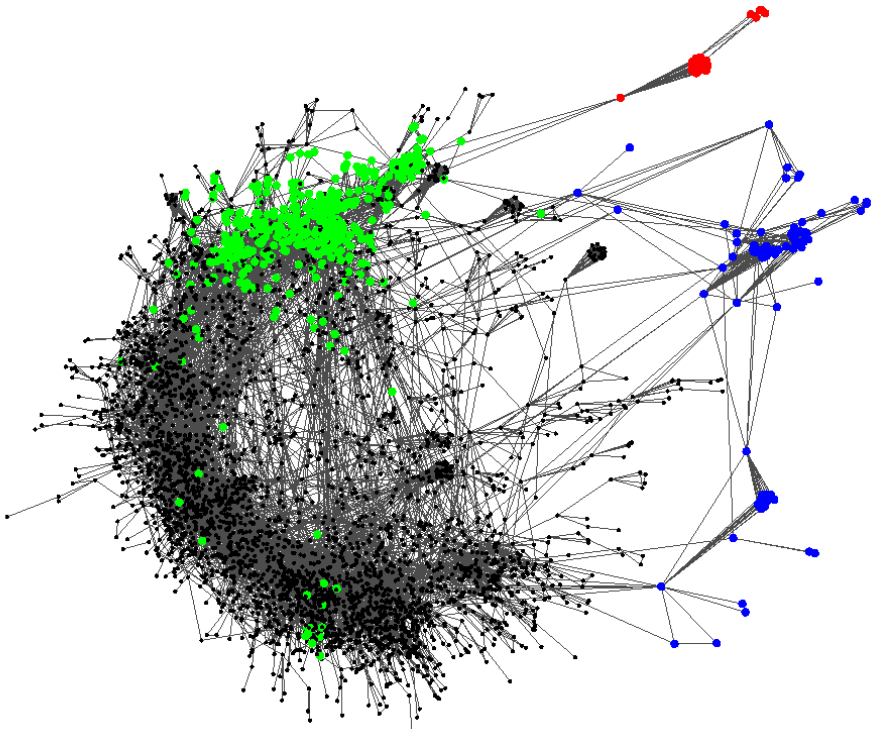
[JMLR '10]

# Small vs. Large Networks

- **Small and large networks are very different:**




$$\Theta = \begin{bmatrix} 0.99 & 0.17 \\ 0.17 & 0.82 \end{bmatrix}$$

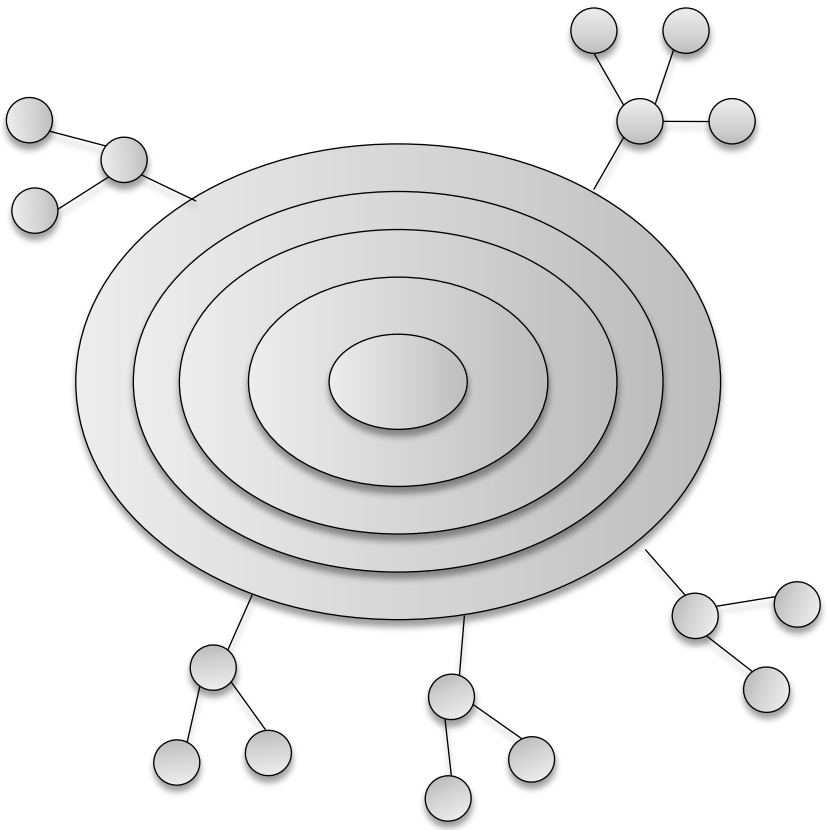


$$\Theta = \begin{bmatrix} 0.99 & 0.54 \\ 0.49 & 0.13 \end{bmatrix}$$

# Implications (1)

**Large scale network structure:**

- **Nested Core-periphery**
  - Recursive onion-like structure of the network where each layer decomposes into a core and periphery



# Implications (2)

- **Remember the SKG theorems:**
  - **Connected**, if $b+c>1$:
    - 0.55+0.15 > 1. **No!**
  - **Giant component**, if $(a+b)\cdot(b+c)>1$:
    - (0.99+0.55)·(0.55+0.15) > 1. **Yes!**

$$\Theta = \begin{bmatrix} 0.99 & 0.55 \\ 0.55 & 0.15 \end{bmatrix}$$

- Real graphs are in the in the parameter region analogous to the giant component of an **extremely sparse $G_{np}$**