Quick Tour of Linear Algebra and Graph Theory

CS224W: Social and Information Network Analysis
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Based on Peter Lofgren, Yu ”Wayne” Wu, and Borja Pelato’s previous versions

Matrices and Vectors

- Matrix: A rectangular array of numbers, e.g., $A \in \mathbb{R}^{m \times n}$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- Vector: A matrix consisting of only one column (default) or one row, e.g., $x \in \mathbb{R}^n$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Linear Function

A linear function M is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ that satisfies two properties:

1. For all $x, y \in \mathbb{R}$,

$$M(x + y) = M(x) + M(y)$$

2. For all $x \in \mathbb{R}$ and all $a \in \mathbb{R}$ (scalar)

$$M(ax) = aM(x)$$

Every linear function can be represented by a matrix. Every matrix is a linear function.

Matrix Multiplication

- If $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, then their product $AB \in \mathbb{R}^{m \times p}$ is the unique matrix such that for any $x \in \mathbb{R}^p$,

$$(AB)(x) = A(B(x)).$$

  - Number of columns of A **must** equal number of rows of B

Matrix Multiplication

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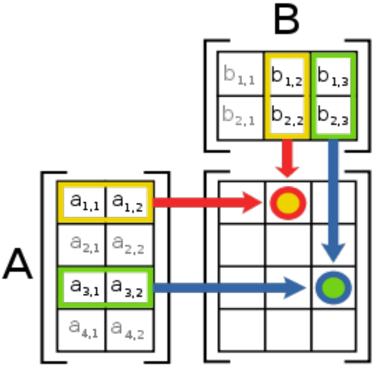
  - Number of columns of A **must** equal number of rows of B
- We can compute the product $C = AB$ using this formula:

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Matrix Multiplication

$$(AB)(x) = A(B(x))$$



Properties of Matrix Multiplication

- Associative: $(AB)C = A(BC)$
- Distributive: $A(B + C) = AB + AC$
- Non-commutative: $AB \neq BA$
  - They don't even have to be the same size!

Operators and properties

- Transpose: $A \in \mathbb{R}^{m \times n}$, then $A^T \in \mathbb{R}^{n \times m}$: $(A^T)_{ij} = A_{ji}$.
  For example, if
  $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
  then
  $$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$
- Properties:
  - $(A^T)^T = A$
  - $(AB)^T = B^T A^T$
  - $(A + B)^T = A^T + B^T$

Identity Matrix

- Identity matrix: $I = I_n \in \mathbb{R}^{n \times n}$:

$$I_{ij} = \begin{cases} 1 & \text{i=j}, \\ 0 & \text{otherwise.} \end{cases}$$

- $\forall A \in \mathbb{R}^{m \times n}$: $AI_n = I_m A = A$

$$I_1 = \begin{bmatrix} 1 \end{bmatrix},\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \cdots,\ I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Diagonal Matrix

- Diagonal matrix: $D = diag(d_1, d_2, \ldots, d_n)$:

$$D_{ij} = \begin{cases} d_i & \text{j=i,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Other Special Matrices

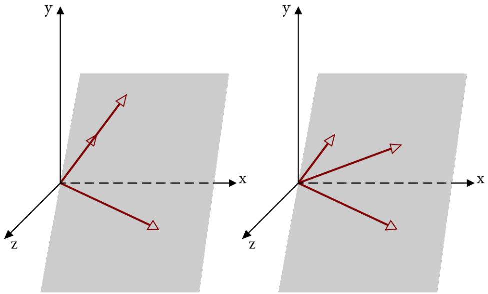
- Symmetric matrices: $A \in \mathbb{R}^{n \times n}$ is symmetric if $A = A^T$.
- Orthogonal matrices: $U \in \mathbb{R}^{n \times n}$ is orthogonal if $UU^T = I = U^T U$
  - Every column is orthogonal to every other column (dot product = 0)

Linear Independence and Rank

- A set of vectors $\{x_1, \ldots, x_n\}$ is linearly independent if $\nexists\{\alpha_1, \ldots, \alpha_n\}$: $\sum_{i=1}^{n} \alpha_i x_i = 0$
  - No vector can be expressed as a linear combination of the other vectors
- Rank: $A \in \mathbb{R}^{m \times n}$, then $rank(A)$ is the maximum number of linearly independent columns (or equivalently, rows)
- Properties:
  - $rank(A) \leq \min\{m, n\}$
  - $rank(A) = rank(A^T)$
  - $rank(AB) \leq \min\{rank(A), rank(B)\}$
  - $rank(A + B) \leq rank(A) + rank(B)$

Example of Linear Dependence

These three vectors are linearly dependent because they all lie in the same plane. The matrix with these three vectors as rows has rank 2.



Rank from row-echelon forms

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} R_2 \to 2r_1 + r_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix} R_3 \to -3r_1 + r_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$R_3 \to r_2 + r_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} R_1 \to -2r_2 + r_1 \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

...or you can just plug it into Matlab or Wolfram Alpha

Matrix Inversion

- If $A \in \mathbb{R}^{n \times n}$, $rank(A) = n$, then the inverse of A, denoted A^{-1} is the matrix that: $AA^{-1} = A^{-1}A = I$
- Properties:
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(A^{-1})^T = (A^T)^{-1}$
- The inverse of an orthogonal matrix is its transpose ($UU^T = I = U^T U$)

Eigenvalues and Eigenvectors

- Given $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{C}$ is an eigenvalue of A with the corresponding eigenvector $x \in \mathbb{C}^n$ ($x \neq 0$) if:

$$Ax = \lambda x$$

For example, if

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

then the vector $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ is an eigenvector with eigenvalue 1, because

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 1 \cdot (-3) \\ 1 \cdot 3 + 2 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

Solving for Eigenvalues/Eigenvectors

- Characteristic Polynomial: If $Ax = \lambda x$ then

$$(A - \lambda I)x = 0$$

so $(A - \lambda I)$ is singular (not full rank), so

$$\det(A - \lambda I) = 0.$$

Thus the eigenvalues are exactly the n possibly complex roots of the degree n polynomial equation $det(A - \lambda I) = 0$. This is known as the characteristic polynomial.

- Once we solve for all λ's, we can plug in to find each corresponding eigenvector.

Eigenvalue/Eigenvector Properties

- Usually eigenvectors are normalized to unit length.
- If A is symmetric, then all the eigenvalues are real
- $tr(A) = \sum_{i=1}^{n} \lambda_i$
- $det(A) = \prod_{i=1}^{n} \lambda_i$

Matrix Eigendecomposition

$A \in \mathbb{R}^{n \times n}$, $\lambda_1, \ldots, \lambda_n$ the eigenvalues, and $x_1, \ldots, x_n$ the eigenvectors. $P = [x_1 | x_2 | \ldots | x_n]$, $D = diag(\lambda_1, \ldots, \lambda_n)$, then:

$$\begin{aligned}
AP &= A\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_k\end{bmatrix} \\
&= \begin{bmatrix}A\mathbf{X}_1 & A\mathbf{X}_2 & \cdots & A\mathbf{X}_k\end{bmatrix} \\
&= \begin{bmatrix}\lambda_1\mathbf{X}_1 & \lambda_2\mathbf{X}_2 & \cdots & \lambda_k\mathbf{X}_k\end{bmatrix} \\
&= \begin{bmatrix}\lambda_1 x_{11} & \lambda_2 x_{21} & \cdots & \lambda_k x_{k1} \\ \lambda_1 x_{12} & \lambda_2 x_{22} & \cdots & \lambda_k x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 x_{1k} & \lambda_2 x_{2k} & \cdots & \lambda_k x_{kk}\end{bmatrix} \\
&= \begin{bmatrix}x_{11} & x_{21} & \cdots & x_{k1} \\ x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k} & x_{2k} & \cdots & x_{kk}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k\end{bmatrix} \\
&= PD,
\end{aligned}$$

Matrix Eigendecomposition

- Therefore, $A = PDP^{-1}$.
- In addition:

  $$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}$$

  By induction, $A^n = PD^nP^{-1}$.
- A special case of Singular Value Decomposition

What is Proof by Induction?

Induction:

1. Show result on base case, associated with $n = k_0$
2. Assume result true for $n = i$. Prove result for $n = i + 1$
3. Conclude result true for all $n \geq k_0$

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1. Show result on base case, associated with $n = k_0$
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Example:
For all natural number n, $1 + 2 + 3 + ... + n = \frac{n*(n+1)}{2}$
Base case: when $n = 1$, $1 = 1$.
Assume statement holds for $n = k$, then
$1 + 2 + 3 + ... + k = \frac{k*(k+1)}{2}$.
We see $1 + 2 + 3 + ... + (k + 1) = \frac{k*(k+1)}{2} + (k + 1) = \frac{(k+1)(k+2)}{2}$.

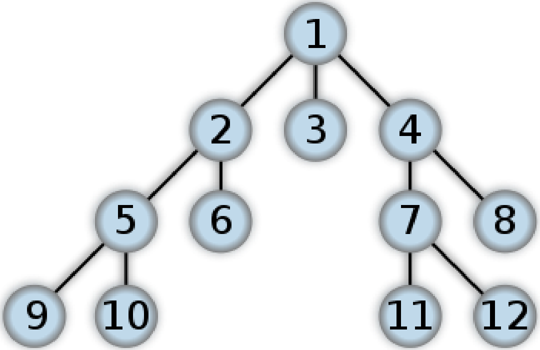
Graph theory

- Definitions: vertex/node, edge/link, loop/cycle, degree, path, neighbor, tree,. . .
- Random graph (Erdos-Renyi): Each possible edge is present independently with some probability p
- (Strongly) connected component: subset of nodes that can all reach each other
- Diameter: longest minimum distance between two nodes
- Bridge: edge connecting two otherwise disjoint connected components

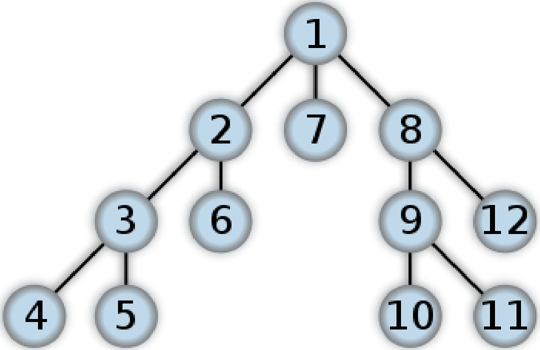
Basic algorithms

- BFS: explore by “layers”
- DFS: go as far as possible, then backtrack
- Greedy: maximize goal at each step
- Binary search: on ordered set, discard half of the elements at each step

Breadth First Search

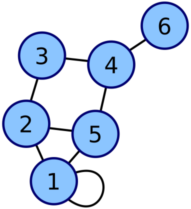


Depth First Search



Adjacency Matrix

The adjacency matrix M of a graph is the matrix such that $M_{i,j} = 1$ if i is connected to j, and $M_{i,j} = 0$ otherwise.



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

For example, this is useful when studying random walks. Renormalize the rows of M so that every row has sum 1. Then if we start at vertex i, after k random walk steps, the distribution of our location is $M^k e_i$, where e_i has a 1 in the ith coordinate and 0 elsewhere.