Preferential Attachment and Network Evolution

CS224W: Social and Information Network Analysis
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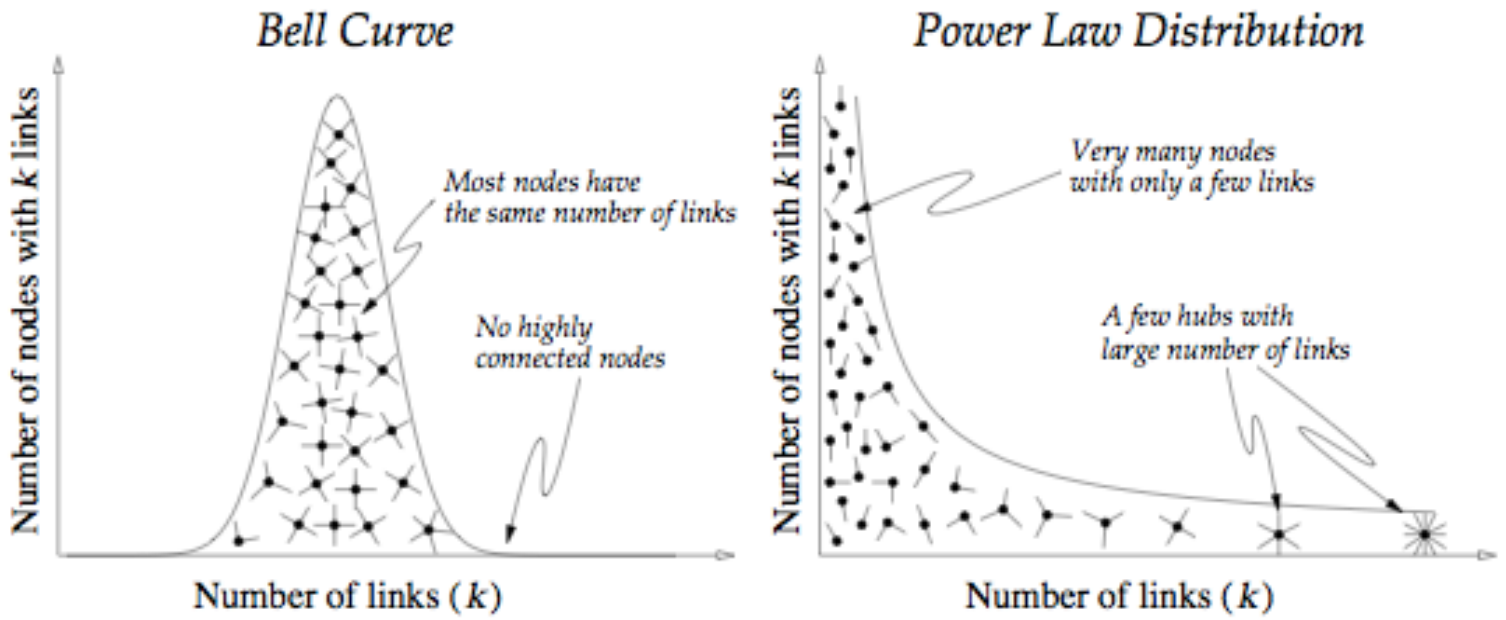
http://cs224w.stanford.edu

Exponential vs. Power-Law Tails

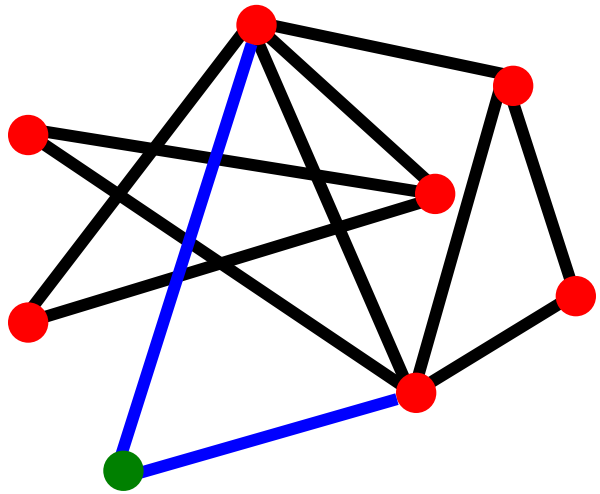


Model: G_{np} ?

Model: Preferential attachment

- **Preferential attachment**
  [Price '65, Albert-Barabasi '99, Mitzenmacher '03]
  - Nodes arrive in order **1,2,…,n**
  - At step j, let d_i be the degree of node $i < j$
  - A new node j arrives and creates m out-links
  - Prob. of j linking to a previous node i is **proportional to degree d_i of node i**

$$P(j \to i) = \frac{d_i}{\sum_k d_k}$$

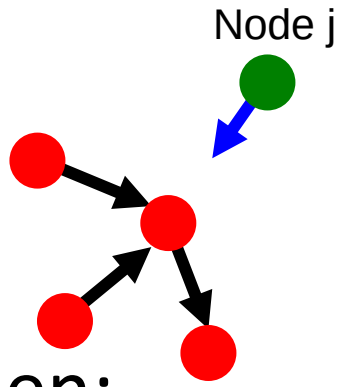


Rich Get Richer

- **New nodes are more likely to link to nodes that already have high degree**
- **Herbert Simon's result:**
  - Power-laws arise from "**Rich get richer**" (**cumulative advantage**)
- **Examples** [Price '65]
  - **Citations:** New citations to a paper are proportional to the number it already has

[Mitzenmacher, '03]

The Exact Model



We will analyze the following model:

- Nodes arrive in order $1,2,3,\ldots,n$
- When node ***j*** is created it makes a **single out-link** to an earlier node $\boldsymbol{i}$ chosen:
  - **1)** With prob. $\boldsymbol{p}$, $\boldsymbol{j}$ links to $\boldsymbol{i}$ chosen **uniformly at random** (from among all earlier nodes)
  - **2)** With prob. $\boldsymbol{1-p}$, node $\boldsymbol{j}$ chooses $\boldsymbol{i}$ uniformly at random and links **to node *l that i* points to**
    - **This is same as saying:** With prob. $\boldsymbol{1-p}$, node $\boldsymbol{j}$ links to node $\boldsymbol{l}$ with prob. proportional to $\boldsymbol{d_l}$ (the in-degree of $\boldsymbol{l}$)
  - **Our graph is directed:** Every node has out-degree **1**

The Model Givens Power-Laws

- **Claim:** The described model generates networks where the fraction of nodes with in-degree k scales as:

$$P(d_i = k) \propto k^{-(1+\frac{1}{q})}$$

where q=1-p

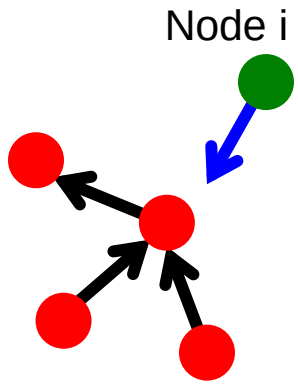
So we get power-law degree distribution with exponent:

$$\alpha = 1 + \frac{1}{1-p}$$

Continuous Approximation

- **Consider deterministic and continuous approximation** to the degree of node $\boldsymbol{i}$ as a function of time $\boldsymbol{t}$
  - $\boldsymbol{t}$ is the number of nodes that have arrived so far
  - In-Degree $\boldsymbol{d_i(t)}$ of node $\boldsymbol{i}$ $(i = 1,2, \ldots, n)$ is a **continuous quantity** and it **grows deterministically** as a function of time $\boldsymbol{t}$
- **Plan: Analyze $\boldsymbol{d_i(t)}$ – continuous in-degree** of node $\boldsymbol{i}$ at time $\boldsymbol{t} > \boldsymbol{i}$

Continuous Degree: What We Know

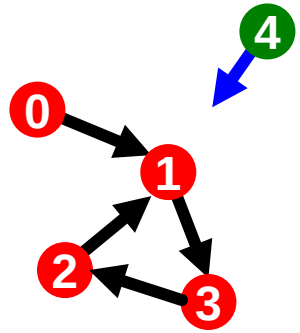


- **Initial condition:**
  - $d_i(t) = 0$, when $t = i$ **(node *i* just arrived)**
- **Expected change of $d_i(t)$ over time:**
  - Node i gains an in-link at step $t + 1$ only if a link from a newly created node $t + 1$ points to it.
  - **What's the probability of this event?**
    - With prob. p node $t + 1$ links **randomly**:
      - Links to our node i with prob. $1/t$
    - With prob. $1 - p$ node $t + 1$ links **preferentially**:
      - Links to our node i with prob. $d_i(t)/t$
  - **Prob. node $t + 1$ links to i is:** $p\frac{1}{t} + (1 - p)\frac{d_i(t)}{t}$

Continuous Degree

- **At $t = 4$ node $i = 4$ comes. It has out-degree of 1 to deterministically share with other nodes:**

Node i	$d_i(t)$	$d_i(t+1)$
0	0	$=0 + p\frac{1}{4} + (1-p)\frac{0}{4}$
1	2	$=2 + p\frac{1}{4} + (1-p)\frac{2}{4}$
2	0	$=0 + p\frac{1}{4} + (1-p)\frac{1}{4}$
3	1	$=1 + p\frac{1}{4} + (1-p)\frac{1}{4}$
4	/	0



- $\boldsymbol{d_i(t) - d_i(t-1) = \frac{\mathrm{d}d_i(t)}{\mathrm{d}t} = \mathrm{p}\frac{1}{t} + (1-p)\frac{d_i(t)}{t}}$
- **How does $d_i(t)$ evolve as $t \to \infty$?**

What is the rate of growth of d_i?

- **Expected change of $d_i(t)$:**
  - $\boldsymbol{d_i(t+1) - d_i(t) = p\frac{1}{t} + (1-p)\frac{d_i(t)}{t}}$
  - $\frac{\mathrm{d}d_i(t)}{\mathrm{d}t} = p\frac{1}{t} + (1-p)\frac{d_i(t)}{t} = \frac{p+qd_i(t)}{t}$ $\quad q = (1-p)$
  - $\frac{1}{p+qd_i(t)}\mathrm{d}d_i(t) = \frac{1}{t}\mathrm{d}t$ $\quad$ Divide by $p + q\ d_i(t)$
  - $\int \frac{1}{p+qd_i(t)}\mathrm{d}d_i(t) = \int \frac{1}{t}\mathrm{d}t$ $\quad$ integrate
  - $\frac{1}{q}\ln\left(p + qd_i(t)\right) = \ln t + c$ $\quad$ Exponentiate and let $A = e^c$
  - $p + qd_i(t) = e^c\ t^q \Rightarrow \boldsymbol{d_i(t) = \frac{1}{q}(At^q - p)}$ $\quad$ **A=?**

What is the constant A?

$$d_i(t) = \frac{1}{q}(At^q - p)$$

What is the value of constant A?

- **We know:** $d_i(i) = 0$
- **So:** $d_i(i) = \frac{1}{q}(Ai^q - p) = 0$
- $\Rightarrow \boldsymbol{A = \frac{p}{i^q}}$
- **And so** $\Rightarrow \boldsymbol{d_i(t) = \frac{p}{q}\left(\left(\frac{t}{i}\right)^q - 1\right)}$

Observation: Old nodes (small i values) have higher in-degrees $d_i(t)$

Degree Distribution

- **What is $F(k)$ the fraction of nodes that has degree less than k at time t?**
  - **How many nodes have degree < k?**
    - $d_i(t) = \frac{p}{q}\left(\left(\frac{t}{i}\right)^q - 1\right) < k$
    - Solve for i and obtain: $i < t\left(\frac{q}{p}k + 1\right)^{-\frac{1}{q}}$
- **There are t nodes total at time t so the fraction $F(k)$ is:**

$$F(k) = \left[\frac{q}{p}k + 1\right]^{-\frac{1}{q}}$$

Degree Distribution

- **What is the fraction of nodes with degree exactly k?**
  - **Take derivative of $F(k)$:**
    - **$F(k)$ is CDF, so $F'(k)$ is the PDF!**

$$F(k) = \left[\frac{q}{p}k + 1\right]^{-\frac{1}{q}}$$

$$F'(k) = \frac{1}{p}\left[\frac{q}{p}k + 1\right]^{-1-\frac{1}{q}} \Rightarrow \alpha = 1 + \frac{1}{1-p}$$

q.e.d.

Preferential attachment: Good news

- **Preferential attachment gives power-law degrees!**
- Intuitively reasonable process
- Can tune p to get the observed exponent
  - On the web, $P[\text{node has degree } d] \sim d^{-2.1}$
  - $2.1 = 1+1/(1-p) \rightarrow p \sim 0.1$

Preferential Attachment: Bad News

- **Preferential attachment is not so good at predicting network structure**
  - **Age-degree correlation**
    - **Solution:** Node fitness (virtual degree)
  - **Links among high degree nodes:**
    - On the web nodes sometime avoid linking to each other
- **Further questions:**
  - **What is a reasonable model for how people sample through network node and link to them?**
    - Short random walks

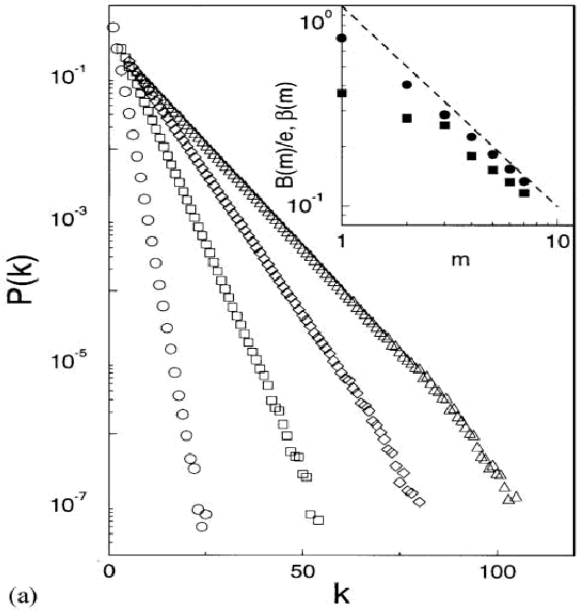
Many models lead to Power-Laws

- **Copying mechanism** (directed network)
  - Select a node and an edge of this node
  - Attach to the endpoint of this edge
- **Walking on a network** (directed network)
  - The new node connects to a node, then to every
  - first, second, … neighbor of this node
- **Attaching to edges**
  - Select an edge and attach to both endpoints of this edge
- **Node duplication**
  - Duplicate a node with all its edges
  - Randomly prune edges of new node

Preferential attachment: Reflections

Extra!

- **Two changes from the G_{np}**
  - The network grows
  - Preferential attachment
- **Do we need both? Yes!**
  - **Add growth to G_{np}** (assume $p = 1$):
    - $X_j =$ degree of node j at the end
    - $X_j(u) =$ 1 if u links to j, else 0
    - $X_j = X_j(j+1) + X_j(j+2) + \cdots + X_j(n)$
    - $E[X_j(u)] = P[u \text{ links to } j] = 1/(u-1)$
    - $E[X_j] = \sum_{j+1}^{n} \frac{1}{u-1} = \frac{1}{j} + \frac{1}{j+1} + \cdots + \frac{1}{n-1} = H_{n-1} - H_j$
    - $E[X_j] = \log(n-1) - \log(j) = \log((n-1)/j)$ NOT $\left(\frac{n}{j}\right)^{\alpha}$



H_n…nth harmonic number:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \approx \log(n)$$

Distances in Preferential Attachment

Extra!

$$\bar{h} = \begin{cases} const & \alpha = 2 \\ \frac{\log\log n}{\log(\alpha - 1)} & 2 < \alpha < 3 \\ \frac{\log n}{\log\log n} & \alpha = 3 \\ \log n & \alpha > 3 \end{cases}$$

Ultra small world

Small world

Avg. path length

Degree exponent

Size of the biggest hub is of order O(N). Most nodes can be connected within two steps, thus the average path length will be independent of the network size.

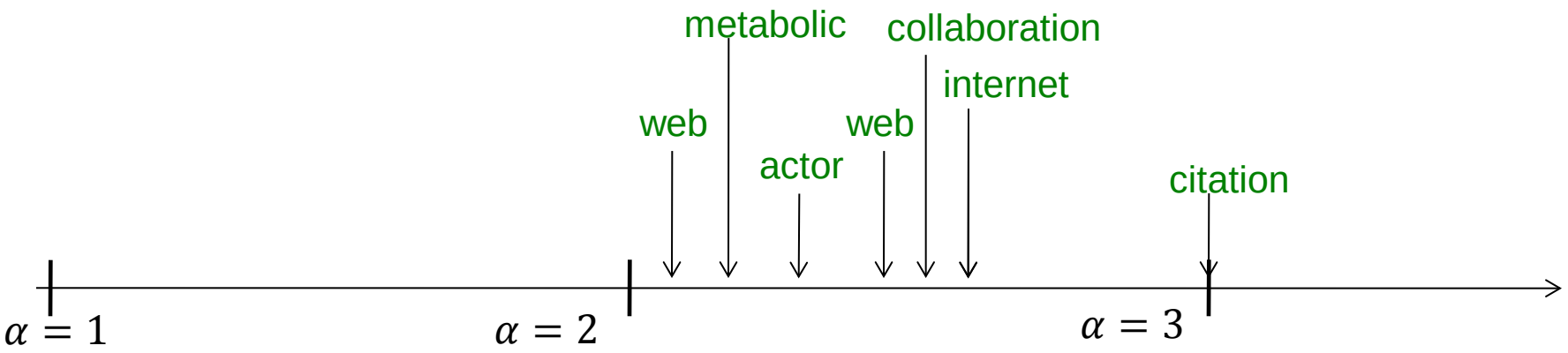
The average path length increases slower than logarithmically. In G_{np} all nodes have comparable degree, thus most paths will have comparable length. In a scale-free network vast majority of the path go through the few high degree hubs, reducing the distances between nodes.

Some models produce $\alpha = 3$. This was first derived by Bollobas et al. for the network diameter in the context of a dynamical model, but it holds for the average path length as well.

The second moment of the distribution is finite, thus in many ways the network behaves as a random network. Hence the average path length follows the result that we derived for the random network model earlier.

Summary: Scale-Free Networks

Extra!

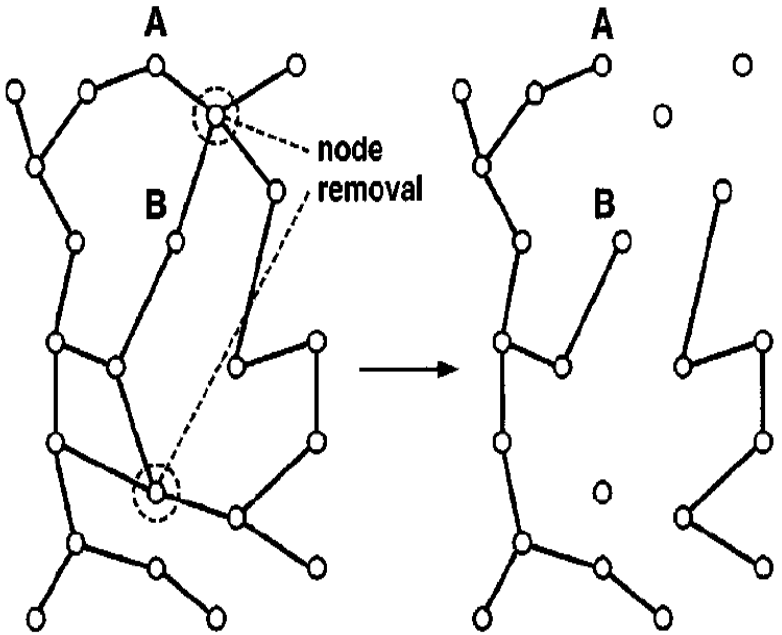


Second moment $\langle k^2 \rangle$ diverges	$\langle k^2 \rangle$ finite
Average $\langle k \rangle$ diverges	$\langle k \rangle$ finite
Ultra small world behavior	Small world

Regime full of anomalies…	The scale-free behavior is relevant	Behaves like a random network

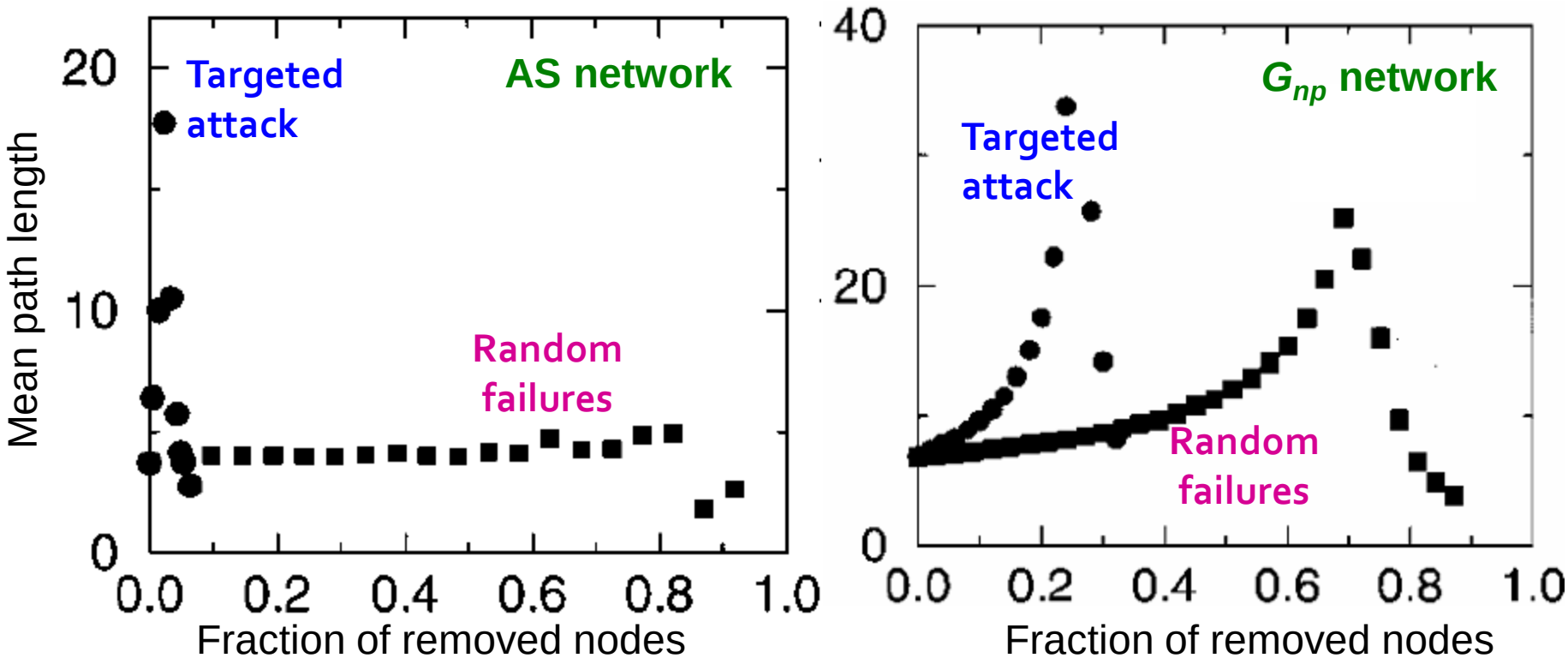
Consequence of Power-Law Degrees

Consequence: Network Resilience



- **How does network connectivity change as nodes get removed?**
  [Albert et al. 00; Palmer et al. 01]
- **Nodes can be removed:**
  - **Random failure:**
    - Remove nodes uniformly at random
  - **Targeted attack:**
    - Remove nodes in order of decreasing degree
- This is important for **robustness of the internet** as well as **epidemiology**

Network Resilience



- **Real networks are resilient to random failures**
- **G_{np} has better resilience to targeted attacks**
  - Need to remove all pages of degree **>5** to disconnect the Web
  - But this is a very small fraction of all web pages

Lessons Learned

- **There is no universal degree exponent characterizing all networks**
- **We need growth and the preferential attachment for the emergence of scale-free property**
  - **The mechanism is domain dependent**
    - Many processes give rise to scale-free networks
- **Modeling real networks:**
  - Identify microscopic processes that occur in the network
  - Measure their frequency from real data
  - Develop dynamical models that capture these processes
  - If the model is correct, it should predict the observations

Evolution of Social Networks

Network Evolution: Observation

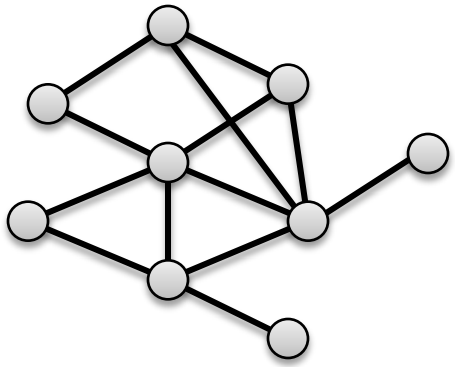
- **Preferential attachment is a model of a growing network**
- **Can we find a more realistic model?**
- **What governs network growth & evolution?**
  - **P1) Node arrival process:**
    - When nodes enter the network
  - **P2) Edge initiation process:**
    - Each node decides when to initiate an edge
  - **P3) Edge destination process:**
    - The node determines destination of the edge

[Leskovec, Backstrom, Kumar, Tomkins, 2008]

[Leskovec et al., KDD '08]

Let's Look at the Data

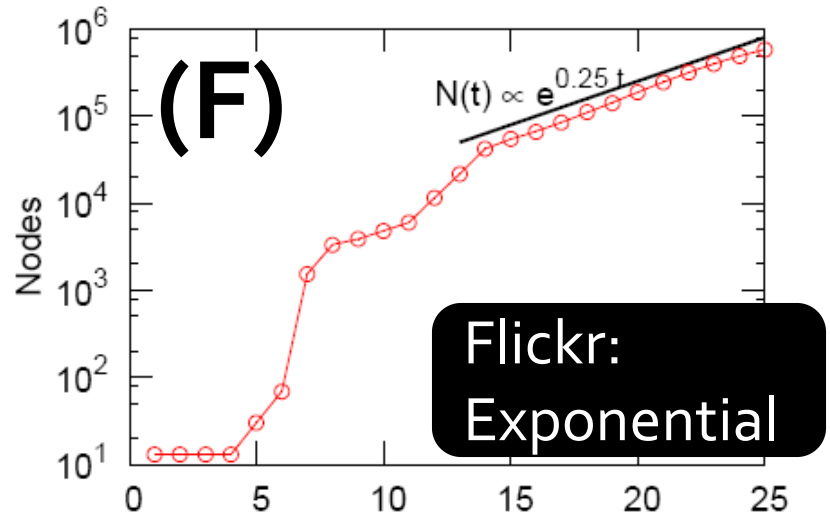
- **4 online social networks with exact edge arrival sequence**
  - For every edge ***(u,v)*** we know **exact time** of the creation t_{uv}
- **Directly observe mechanisms leading to global network properties**

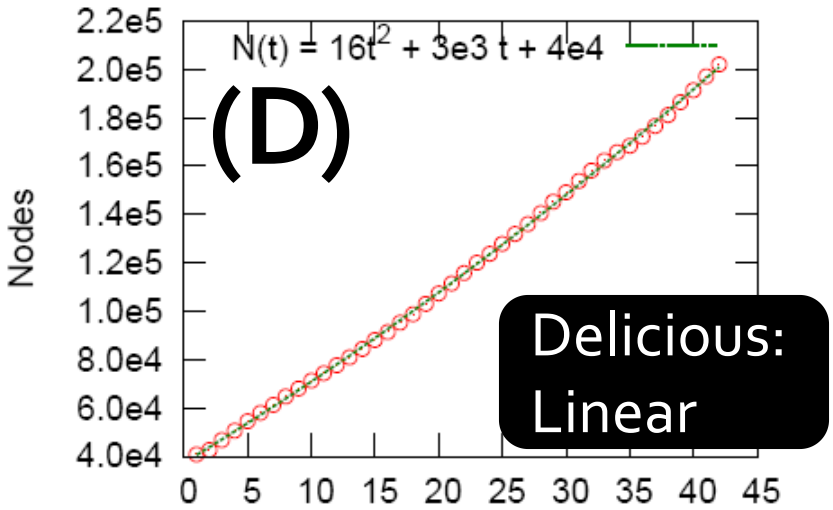


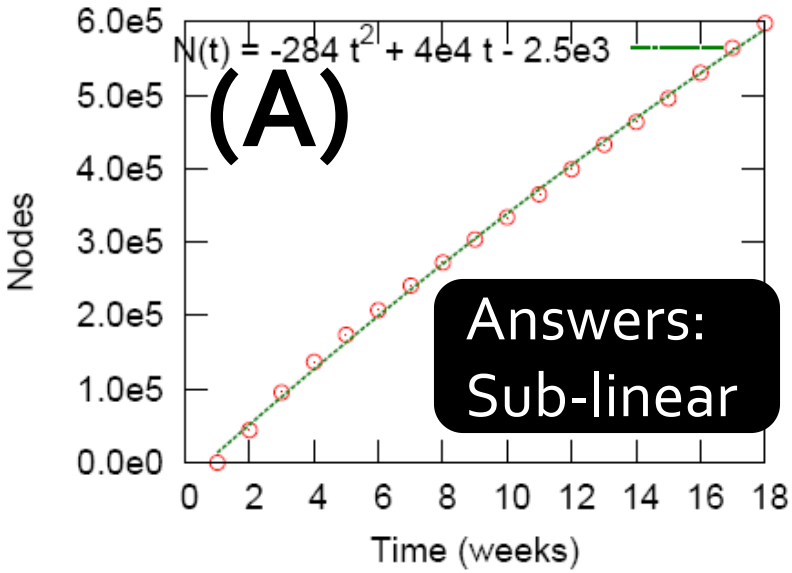
and so on for millions...

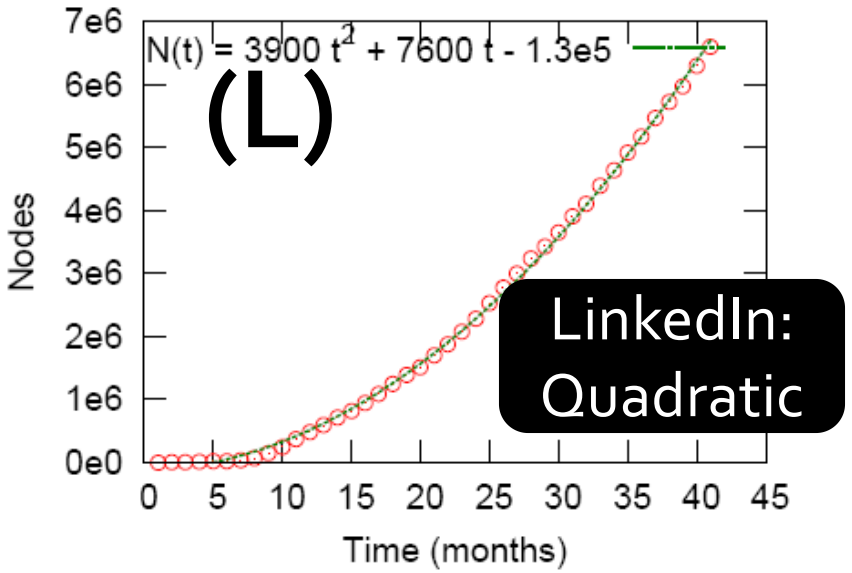
Network	T	N	E
(F) FLICKR (03/2003–09/2005)	621	584,207	3,554,130
(D) DELICIOUS (05/2006–02/2007)	292	203,234	430,707
(A) ANSWERS (03/2007–06/2007)	121	598,314	1,834,217
(L) LINKEDIN (05/2003–10/2006)	1294	7,550,955	30,682,028

P1) When are New Nodes Arriving?



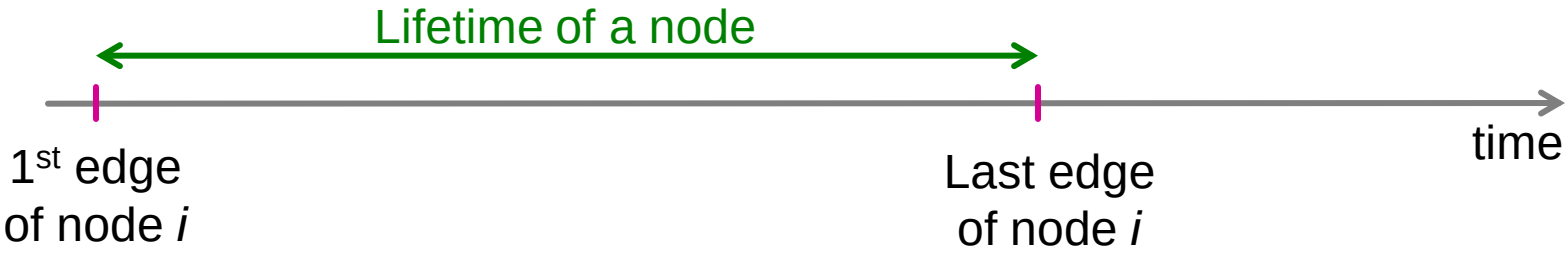




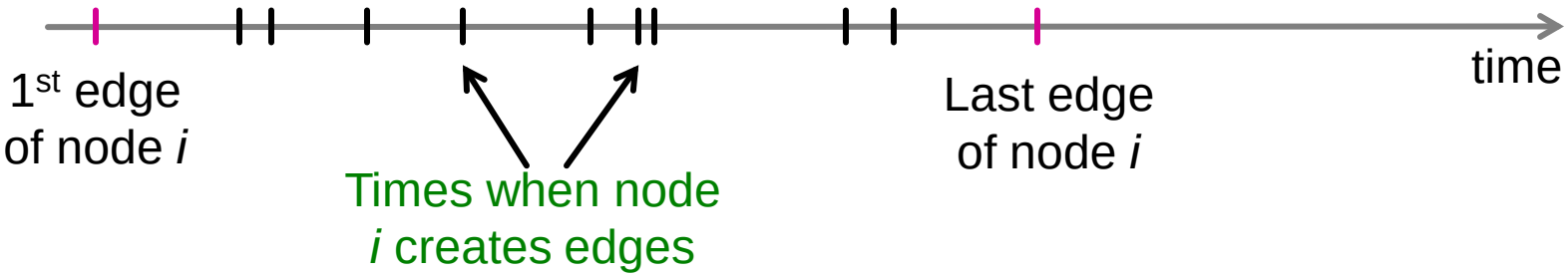


P2) When Do Nodes Create Edges?

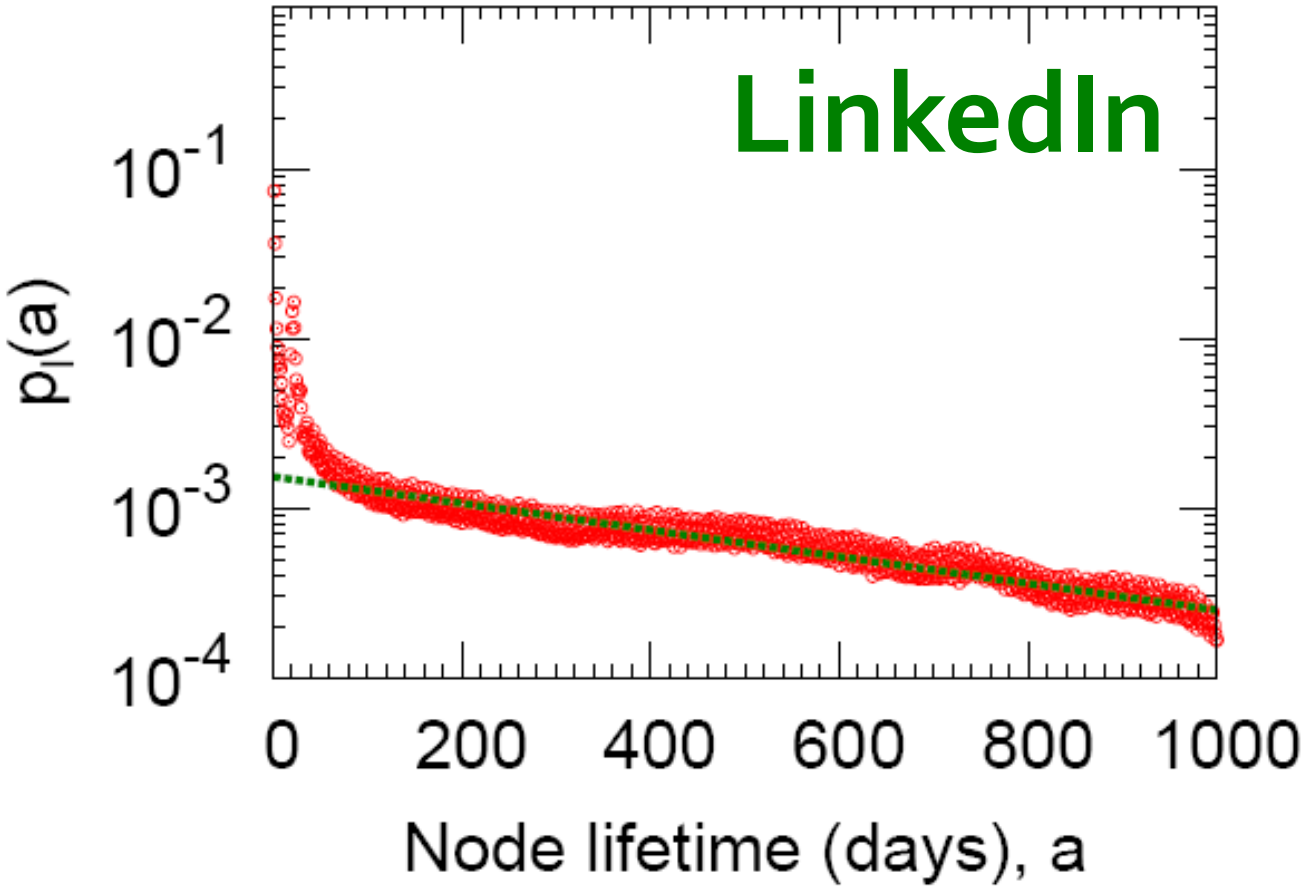
- **How long do nodes live?**
  - Node life-time is the time between the 1st and the last edge of a node



- **How do nodes "wake up" to create links?**



P2) What is Node Lifetime?



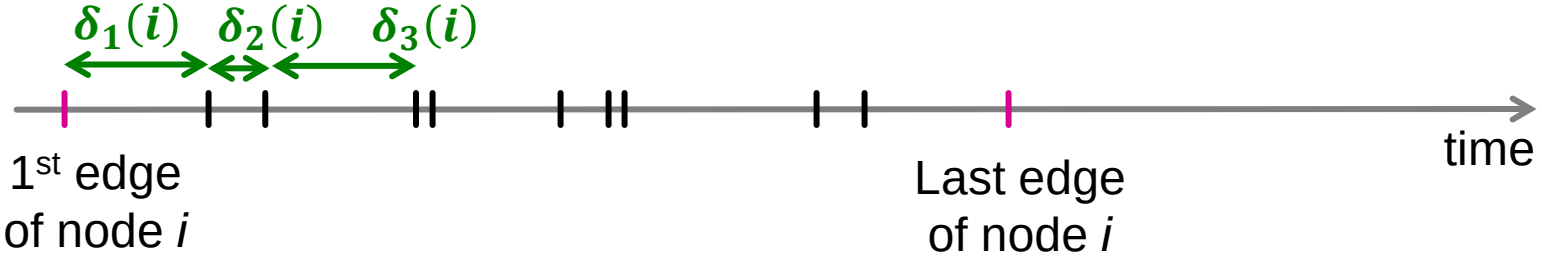
- **Lifetime *a*:** Time between node's first and last edge

Node lifetime is **exponentially distributed**:

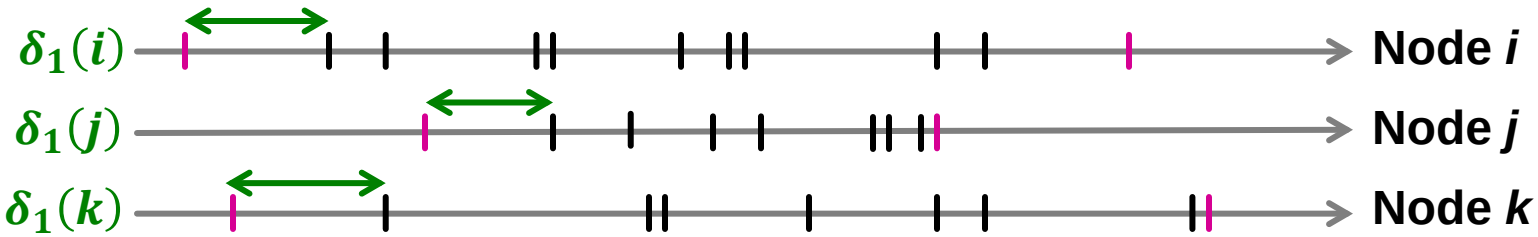
$$p_l(a) = \lambda e^{-\lambda a}$$

P2) When do Nodes Create Edges?

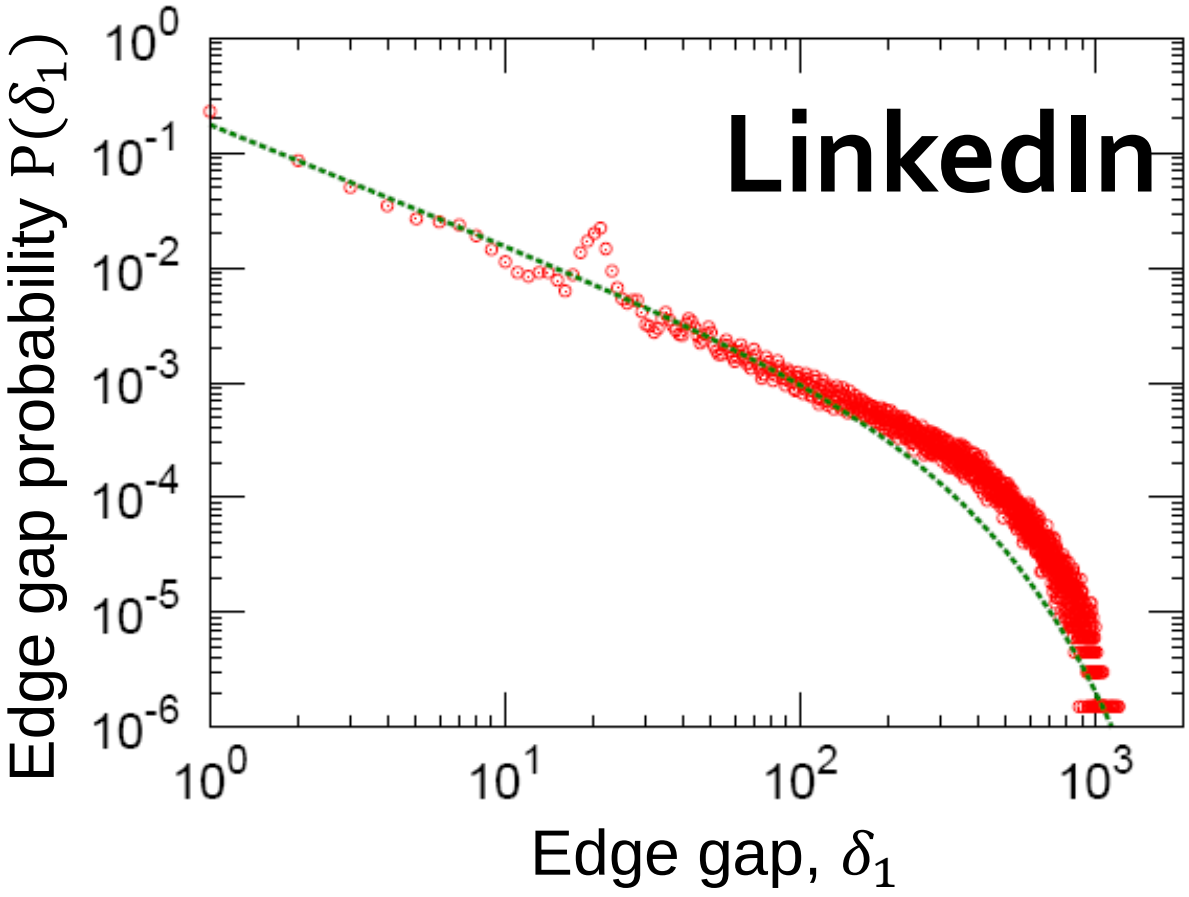
- **How do nodes "wake up" to create edges?**
  - **Edge gap $\delta_d(i)$**: time between d^{th} and $d+1^{st}$ edge of node i:
    - Let $t_d(i)$ be the creation time of d-th edge of node i
    - $\delta_d(i) = t_{d+1}(i) - t_d(i)$



  - **δ_d is a distribution (histogram) of $\delta_d(i)$ over all nodes i**



P2) When do Nodes Create Edges?



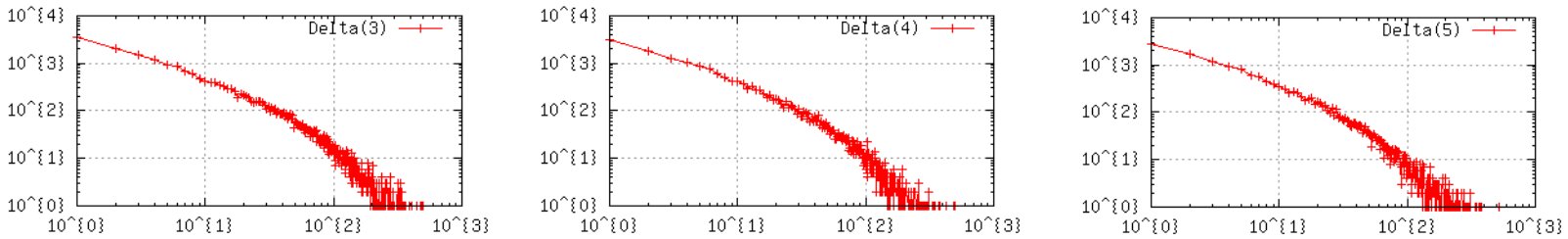
Edge gap δ_d: inter-arrival time between d^{th} and $d+1^{st}$ edge **is distributed by a power-law with exponential cut-off**

For every d we make a separate histogram

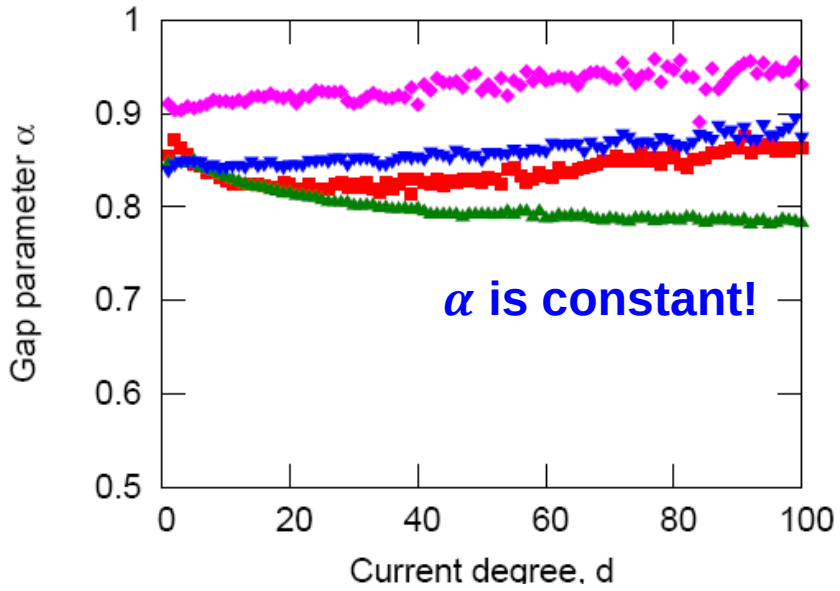
$$p_g(\delta_1) \propto \delta_1^{-\alpha} e^{-\beta}$$

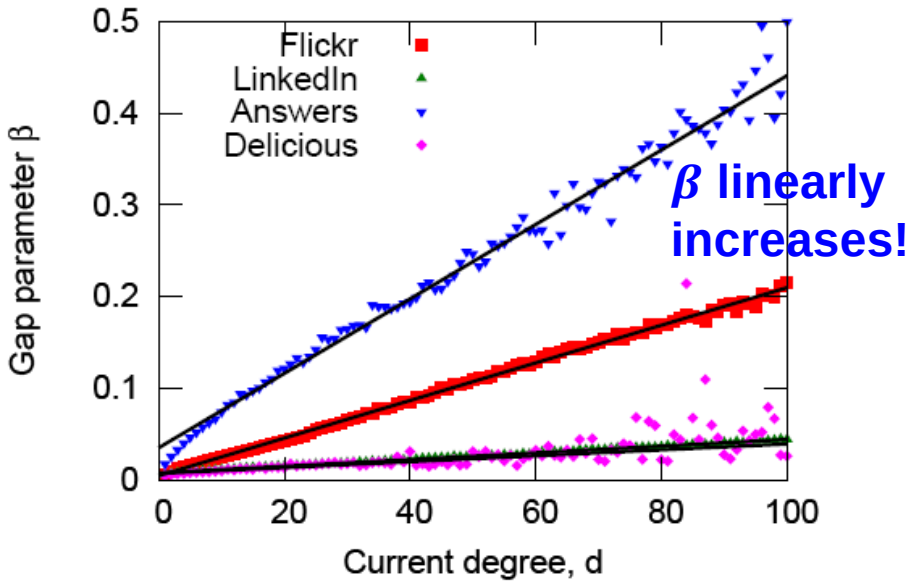
P2) How Do α & β Evolve With d?

- How do α and β change as a function of d?



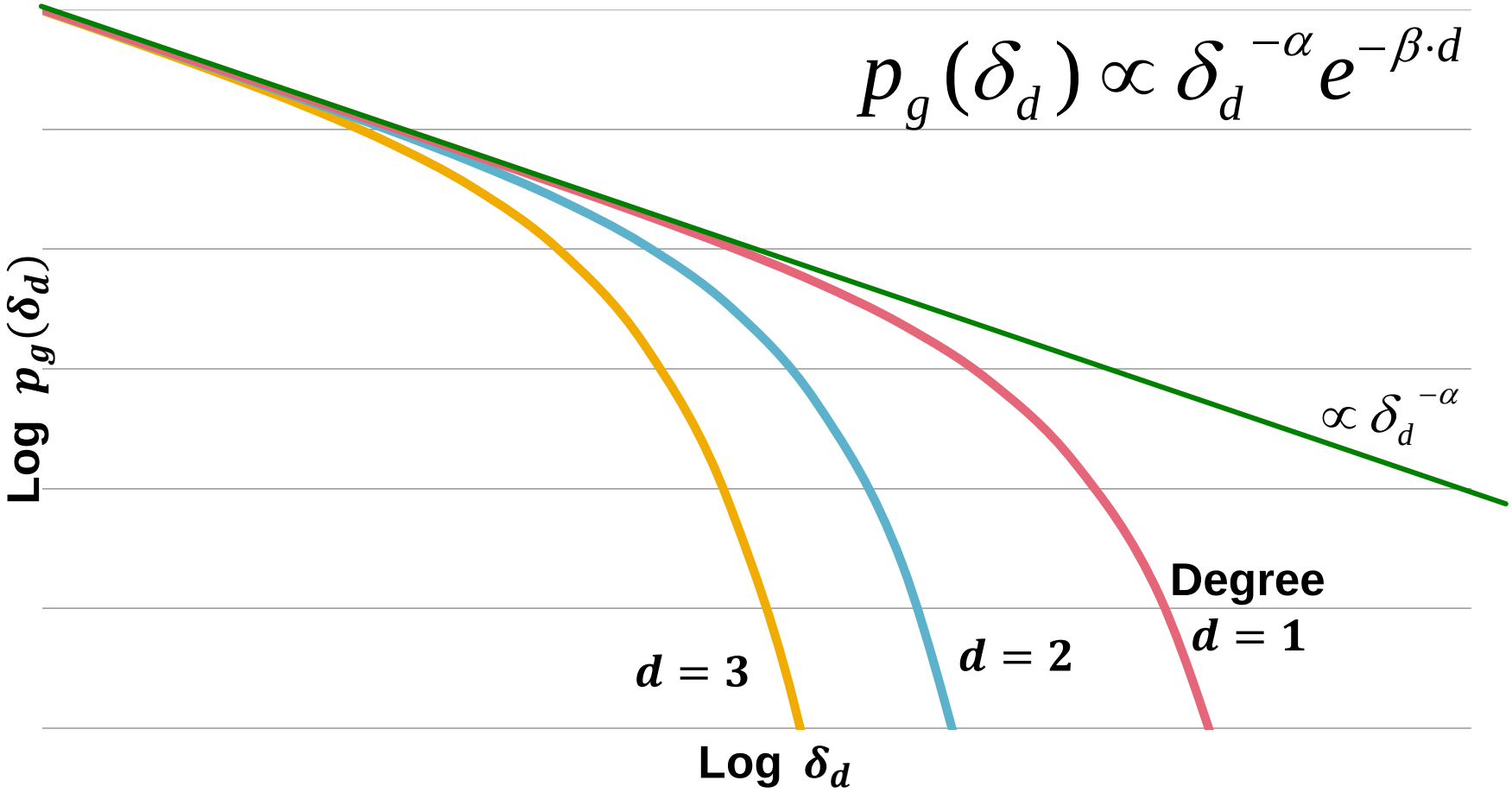
To each plot of δ_d fit: $p_g(\delta_d) \propto \delta_d^{-\alpha_d} e^{-\beta_d}$





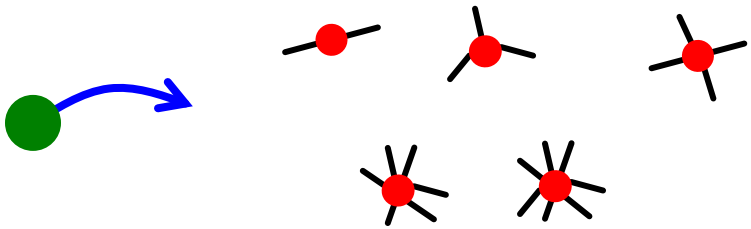
P2) Evolution of Edge Gaps

- α const., β linear in d. What does this mean?
- Gaps get smaller with d!

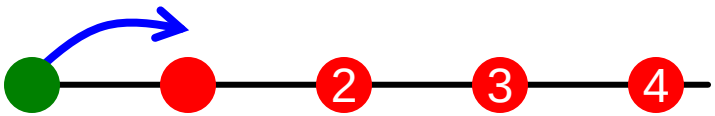


P3) How to Select Destination?

- Source node *i* wakes up and creates an edge
- How does *i* select a target node *j*?
  - What is the degree of the target *j*?
    - Does preferential attachment really hold?



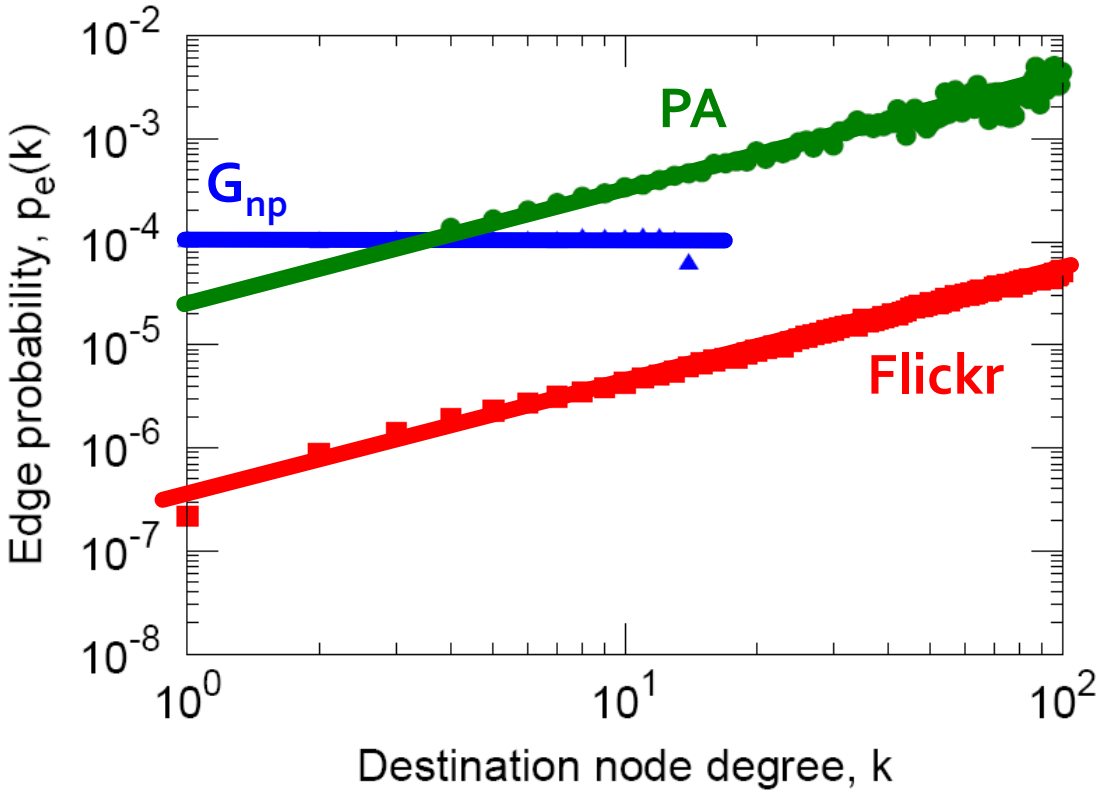
  - How many hops away is the target *j*?
    - Are edges attaching locally?



[Leskovec et al., KDD '08]

Edge Attachment Degree Bias

- **Are edges more likely to connect to higher degree nodes? YES!**



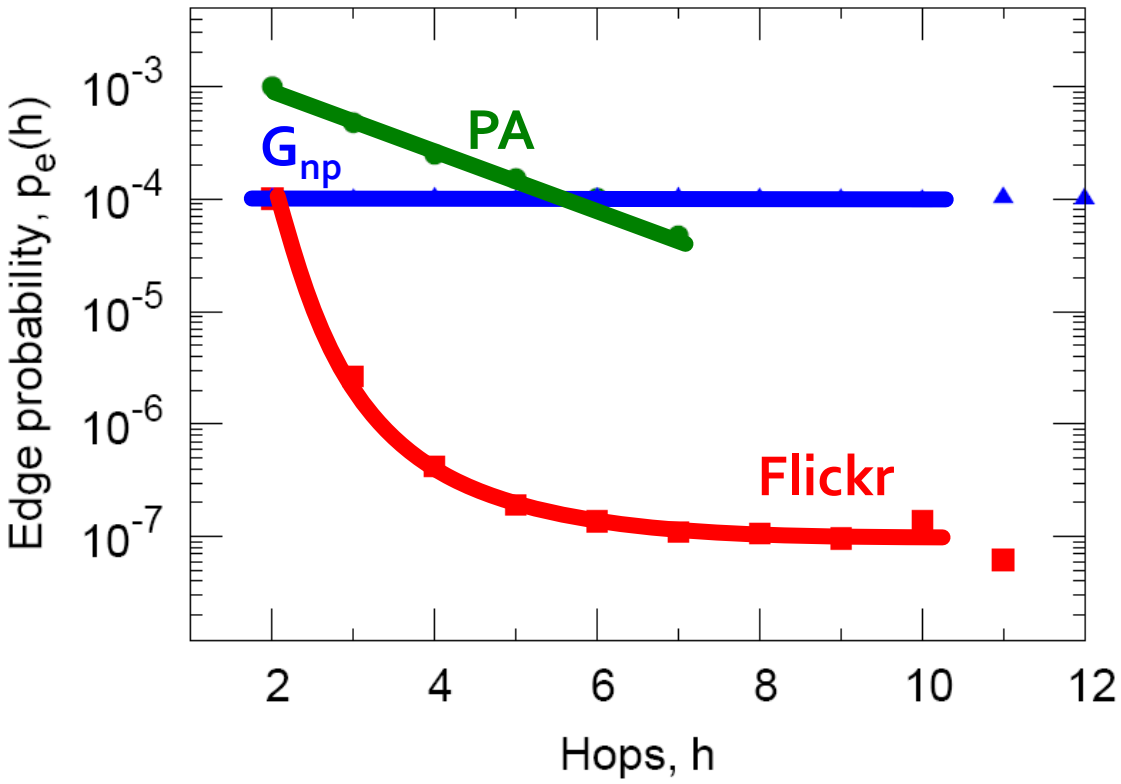
$$p_e(k) \propto k^{\tau}$$

Network	τ
G_{np}	0
PA	1
Flickr	1
Delicious	1
Answers	0.9
LinkedIn	0.6

[Leskovec et al., KDD '08]

How "far" is the Target Node?

- **Just before the edge *(u,w)* is placed how many hops are between *u* and *w*?**



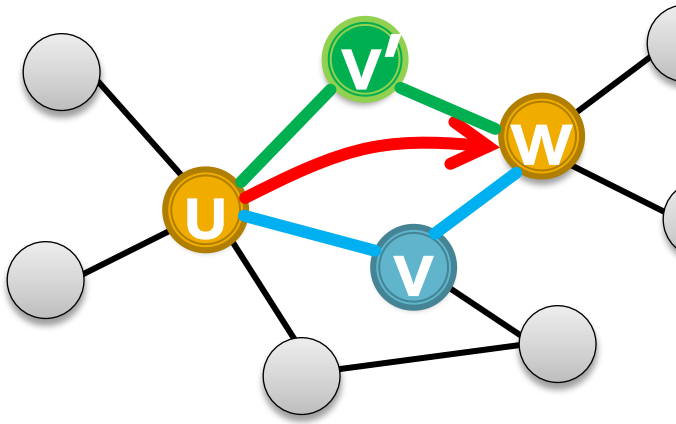
Fraction of triad closing edges	
Network	**% Δ**
Flickr	66%
Delicious	28%
Answers	23%
LinkedIn	50%

Real edges are local!
Most of them close **triangles**!

[Leskovec et al., KDD '08]

How to Close the Triangles?

- **Focus only on triad-closing edges**
- **New triad-closing edge *(u,w)* appears next**
- **2 step walk model:**
  - **u is about to create an edge**
  1. ***u*** choses neighbor ***v***
  2. ***v*** choses neighbor ***w*** and **u** connect**s** to ***w***



- One can use different strategies for choosing **v** and **w: Random-Random works well. Why?**
  - More common friends (more paths) helps
  - High-degree nodes are more likely to be hit

Triad Closing Strategies

[Leskove

Extra!

- **Improvement in log-likelihood over baseline:**
  - **Baseline:** Pick a random node 2 hops away

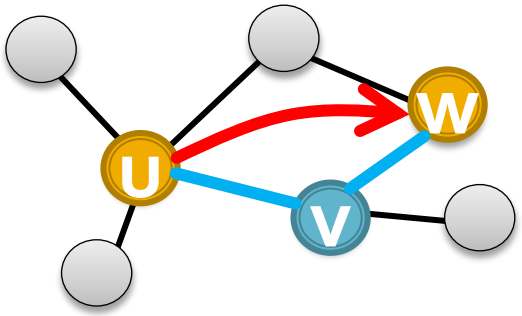
Strategy to select *v* (1st node)

Select *w* (2nd node)

FLICKR	random	$\deg^{0.2}$	com	$\text{last}^{-0.4}$	$\text{comlast}^{-0.4}$
random	13.6	13.9	14.3	16.1	15.7
$\deg^{0.1}$	13.5	14.2	13.7	16.0	15.6
$\text{last}^{0.2}$	14.7	15.6	15.0	17.2	**16.9**
com	11.2	11.6	11.9	13.9	13.4
$\text{comlast}^{0.1}$	11.0	11.4	11.7	13.6	13.2

Strategies to pick a neighbor:

- **random**: uniformly at random
- **deg**: proportional to its degree
- **com**: prop. to the number of common friends
- **last**: prop. to time since last activity
- **comlast**: prop. to **com*last**



[Leskovec et al., KDD '08]

Summary of the Model

- **The model of network evolution**

Process	Model
P1) Node arrival	• Node arrival function is given
P2) Edge initiation	• Node lifetime is exponential • Edge gaps get smaller as the degree increases
P3) Edge destination	Pick edge destination using random-random