Basic Network Properties and the Random Graph Model

CS224W: Social and Information Network Analysis

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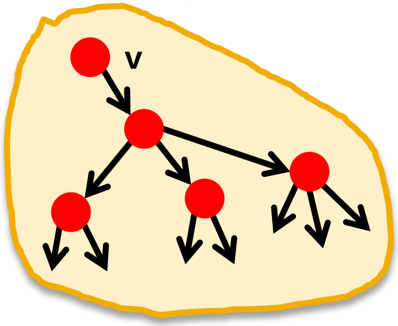
http://cs224w.stanford.edu

Announcement: Recitations

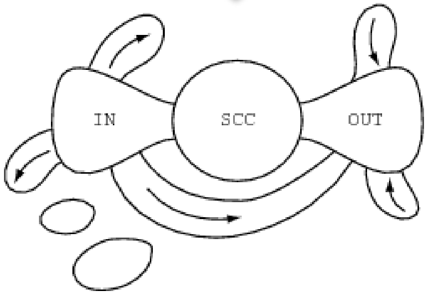
- **Review of basic probability:**
  - Today, Thu 9/29
  - In Gates B01, 4-6pm
- **Review of basic linear algebra:**
  - Tomorrow, Fri 9/30
  - Gates B03, 4-6pm
- **Next week:**
  - Intro to SNAP (Gates B01, 4-6pm on Thu 10/6)
  - Intro to NetworkX (Gates B03, 4-6pm on Fri 10/7)

Structure of Networks

- **Recall from the last lecture:**
  - 1) We took a real system: **the Web**
  - 2) We represented it as a **directed graph**
  - 3) We used the language of graph theory
    - **Strongly Connected Components**
  - 4) We designed a **computational experiment:**
    - Find In- and Out-components of a given node *v*
  - **5) We learned something about the structure of the Web**
- This class:
  - Define basic terminology and measures that you can compute on networks



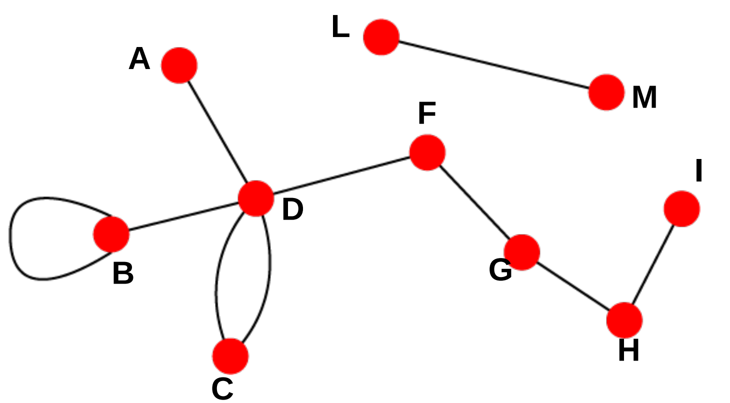
Out(v)



Undirected vs. Directed Networks

Undirected

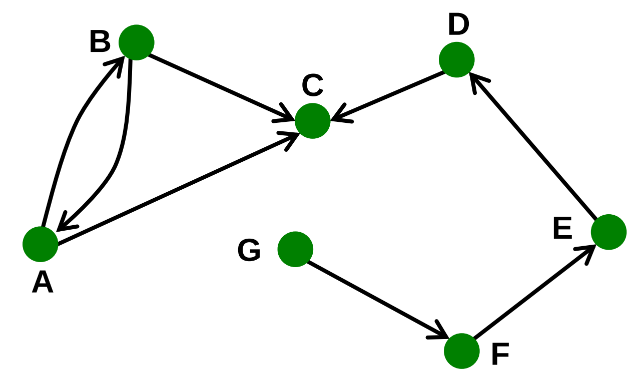
- Links: undirected (symmetrical)



- Undirected links:
  - Collaborations
  - Friendship on Facebook

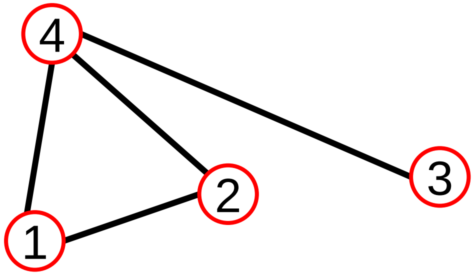
Directed

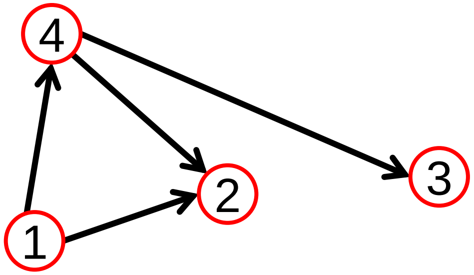
- Links: directed (arcs)



- Directed links:
  - Phone calls
  - Following on Twitter

Adjacency Matrix





A_{ij}=1 if there is a link between node *i* and *j*

A_{ij}=0 if nodes *i* and *j* are not connected to each other

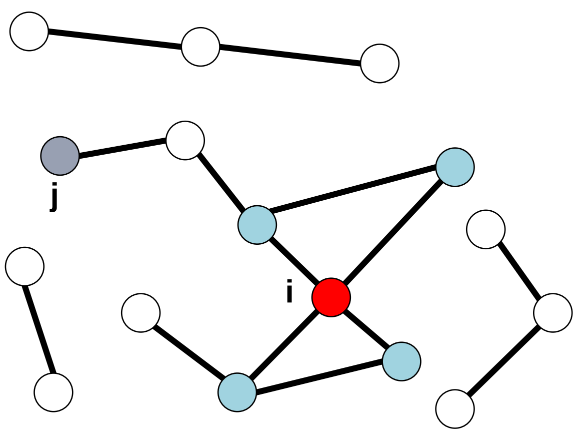
$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Note that for a directed graph (right) the matrix is not symmetric.

Node Degrees

Undirected

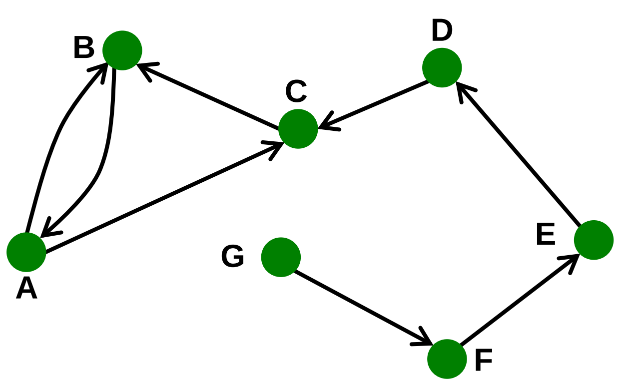


Node degree: the number of links connected to the node

$$k_i = 4$$

Avg. degree: $$\bar{k} \equiv \frac{1}{N}\sum_{i=1}^{N} k_i = \frac{2E}{N}$$

Directed



Source: A node with $k^{in} = 0$

Sink: A node with $k^{out} = 0$

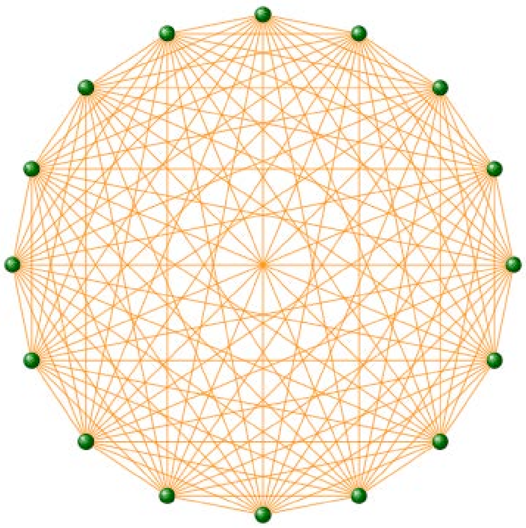
In directed networks we define an in-degree and out-degree.

The (total) degree of a node is the sum of in- and out-degree.

$$k_C^{in} = 2 \qquad k_C^{out} = 1 \qquad k_C = 3$$

$$\bar{k} = \frac{E}{N} \qquad \overline{k^{in}} = \overline{k^{out}}$$

Complete Graph



The maximum number of edges in an undirected graph on N nodes is

$$E_{\max} = \binom{N}{2} = \frac{N(N-1)}{2}$$

A graph with the number of edges $E = E_{max}$ is a **complete graph**,
and its average degree is N-1

Networks are Sparse Graphs

Most real-world networks are sparse

$$E \ll E_{max} \quad (\text{or } \bar{k} \ll N-1)$$

WWW (Stanford-Berkeley):	N=319,717	$\langle k \rangle$=9.65
Social networks (LinkedIn):	N=6,946,668	$\langle k \rangle$=8.87
Communication (MSN IM):	N=242,720,596	$\langle k \rangle$=11.1
Coauthorships (DBLP):	N=317,080	$\langle k \rangle$=6.62
Internet (AS-Skitter):	N=1,719,037	$\langle k \rangle$=14.91
Roads (California):	N=1,957,027	$\langle k \rangle$=2.82
Protein (S. Cerevisiae):	N=1,870	$\langle k \rangle$=2.39

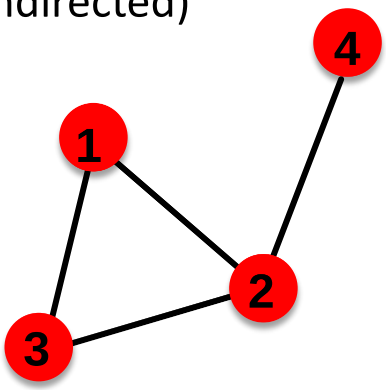
(Source: *Leskovec et al., Internet Mathematics, 2009*)

Consequence: Adjacency matrix is filled with zeros!

(Density (E/N^2): WWW=1.51×10^{-5}, MSN IM = 2.27×10^{-8})

More Types of Graphs:

- **Unweighted**
  (undirected)



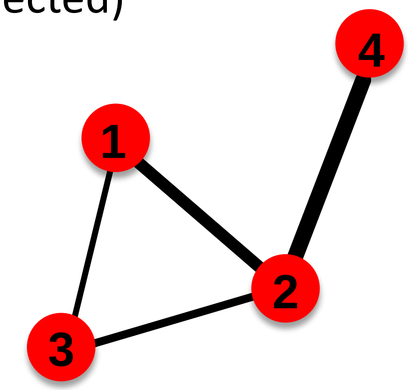
$$A_{ij} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2}\sum_{i,j=1}^{N} A_{ij} \qquad \overline{k} = \frac{2E}{N}$$

Friendships, WWW

- **Weighted**
  (undirected)



$$A_{ij} = \begin{pmatrix} 0 & 2 & 0.5 & 0 \\ 2 & 0 & 1 & 4 \\ 0.5 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

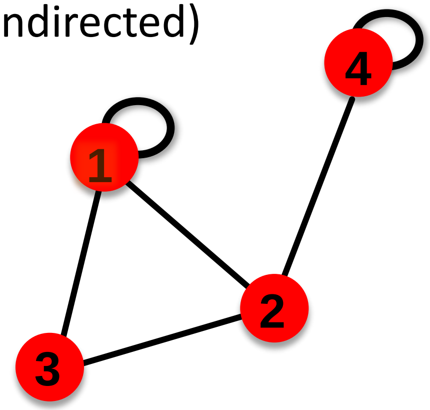
$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2}\sum_{i,j=1}^{N} nonzero(A_{ij}) \qquad \overline{k} = \frac{2E}{N}$$

Call graph, Email graph

More Types of Graphs:

- **Self-edges** (undirected)



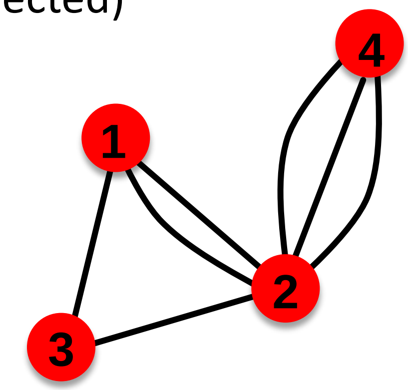
$$A_{ij} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$A_{ii} \neq 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2} \sum_{i,j=1, i \neq j}^{N} A_{ij} + \sum_{i=1}^{N} A_{ii} \qquad ?$$

WWW, Email

- **Multigraph** (undirected)



$$A_{ij} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$E = \frac{1}{2} \sum_{i,j=1}^{N} nonzero(A_{ij}) \qquad \bar{k} = \frac{2E}{N}$$

Social networks, collaboration networks

Network Representations

WWW >> directed multigraph with self-interactions

Facebook friendships >> undirected, unweighted

Citation networks >> unweighted directed acyclic

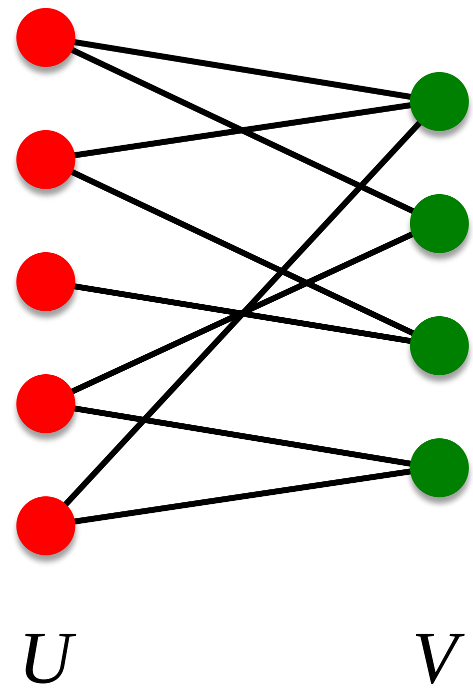
Collaboration networks >> undirected multigraph or weighted

Mobile phone calls >> directed, (weighted?) multigraph

Protein Interactions >> undirected, unweighted with self-interactions

Bipartite Graph

- **Bipartite graph** is a graph whose nodes can be divided into two disjoint sets U and V such that every link connects a node in U to one in V; that is, U and V are independent sets.



- **Examples:**
  - Authors-to-papers
  - Movies-to-Actors
  - Users-to-Movies
- **"Folded" networks**
  - Author collaboration networks
  - Actor collaboration networks

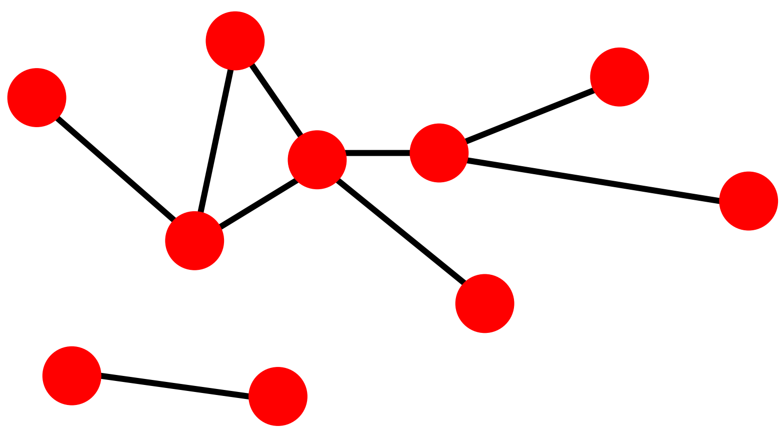
Network Properties:
How to Characterize a Network?

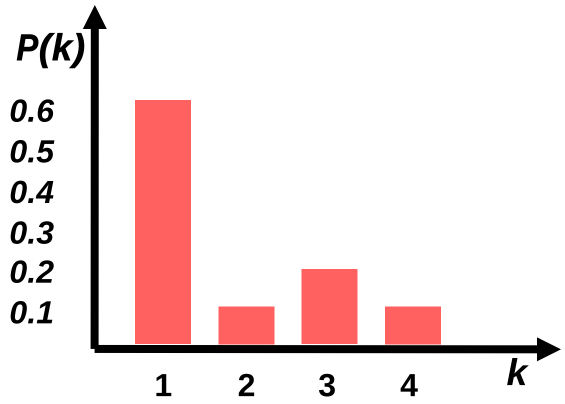
Degree Distribution

- **Degree distribution $P(k)$:** Probability that a randomly chosen node has degree k

  N_k = # nodes with degree k

  $P(k) = N_k / N$ → **plot**



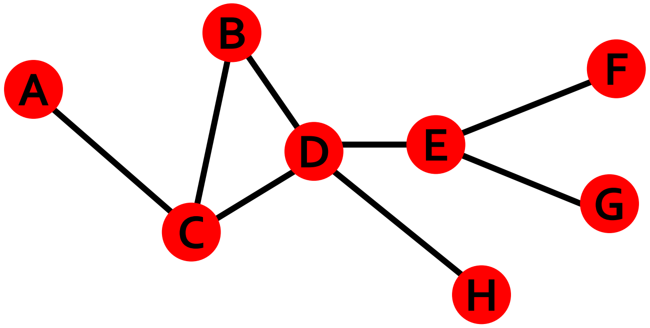


Paths in a Graph

- A ***path*** is a sequence of nodes in which each node is adjacent to the next one

$$P_n = \{i_0, i_1, i_2, \ldots, i_n\} \qquad P_n = \{(i_0, i_1), (i_1, i_2), (i_2, i_3), \ldots, (i_{n-1}, i_n)\}$$

- Path can intersect itself and pass through the same edge multiple times
  - E.g.: ACBDCDEG
  - In a directed graph a path can only follow the direction of the "arrow"



Number of Paths

- **Number of paths between nodes u and v :**
  - **Length $h=1$:** If there is a link between u and v, $A_{uv}=1$ else $A_{uv}=0$
  - **Length $h=2$:** If there is a path of length two between u and v then $A_{uk}A_{kv}=1$ else $A_{uk}A_{kv}=0$

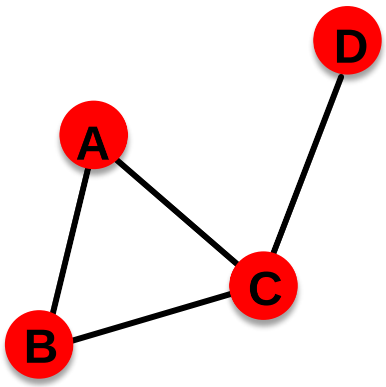
$$H_{uv}^{(2)} = \sum_{k=1}^{N} A_{uk} A_{kv} = [A^2]_{uv}$$

  - **Length h:** If there is a path of length h between u and v then $A_{uk} \,\ldots\, A_{kv}=1$ else $A_{uk} \,\ldots\, A_{kv}=0$
    So, the no. of paths of length h between u and v is

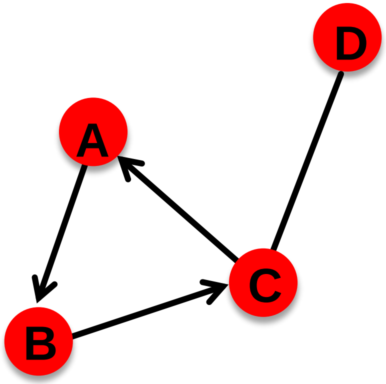
$$H_{uv}^{(h)} = [A^h]_{uv}$$

(holds for both directed and undirected graphs)

Distance in a Graph



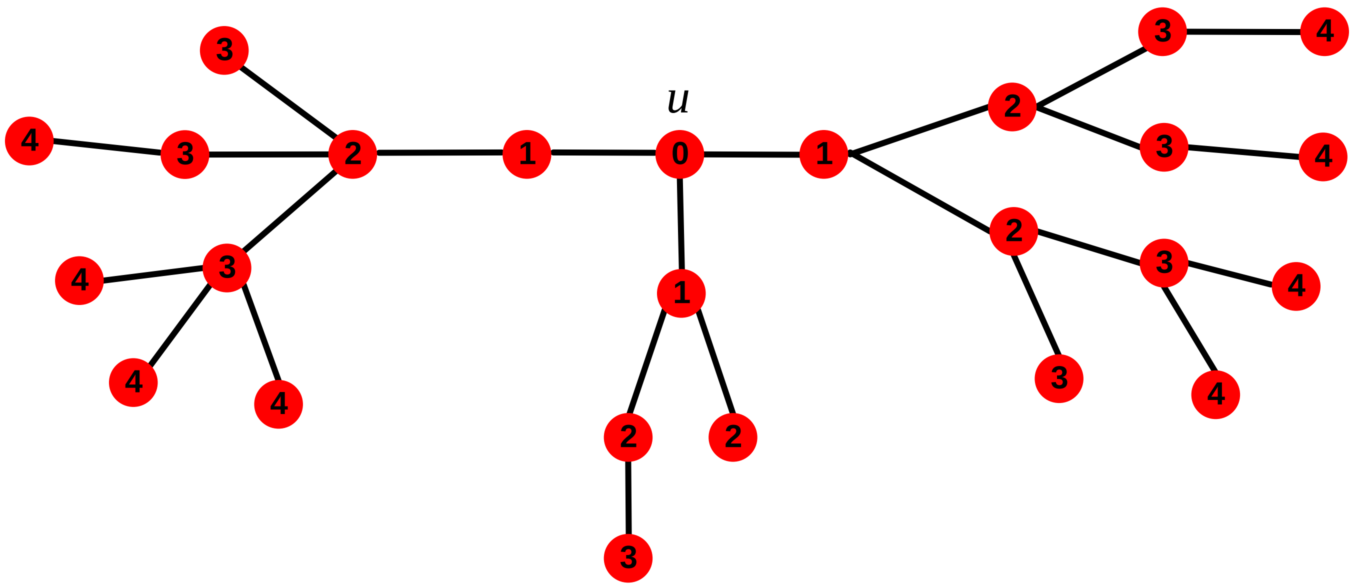
- **Distance (shortest path, geodesic)** between a pair of nodes is defined as the number of edges along the shortest path connecting the nodes.
  - *If the two nodes are disconnected, the distance is defined as infinite



- In **directed graphs** paths need to follow the direction of the arrows.
  - Consequence: Distance is not symmetric: $h(A,C) \neq h(C,A)$

Finding Shortest Paths

- **Breath-First Search:**
  - Start with node u, mark it to be at distance $h_u(u)=0$, add u to the queue
  - While queue not empty:
    - Take node v off the queue, put it's unmarked neighbor w into the queue and mark $h_u(w)=h_u(v)+1$



Network Diameter

- **Diameter:** the maximum (shortest path) distance between any pair of nodes in the graph
- **Average path length/distance** for a connected graph (component) or a strongly connected (component of a) directed graph

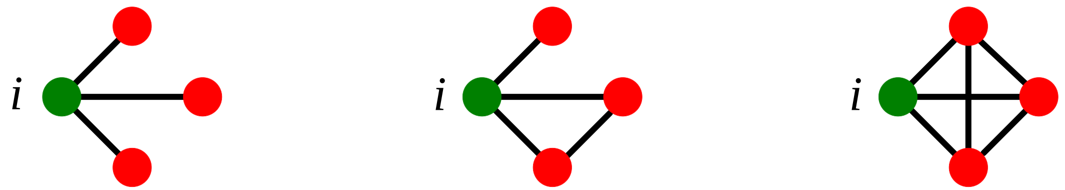
$$\bar{h} = \frac{1}{2E_{\max}} \sum_{i,j \neq i} h_{ij}$$

where h_{ij} is the distance from node i to node j

  - Many times we compute the average only over the connected pairs of nodes (*i.e.*, we ignore "infinite" paths)

Clustering Coefficient

- **Clustering coefficient:**
  - What portion of i's neighbors are connected?
  - Node i with degree k_i
  - $C_i \in [0,1]$
  - $C_i = \frac{2e_i}{k_i(k_i-1)}$ where e_i is the number of edges between the neighbors of node i

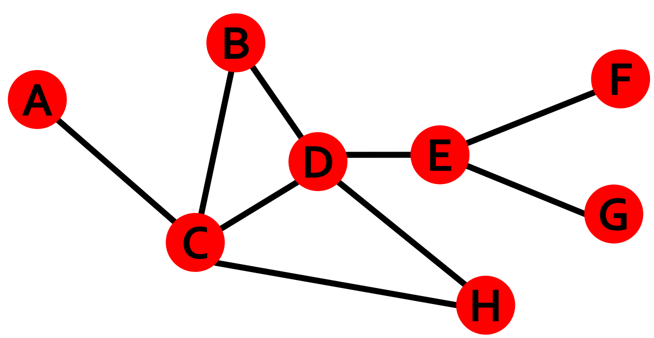


C_i=0 C_i=1/3 C_i=1

- **Average Clustering Coefficient:** $C = \frac{1}{N}\sum_{i}^{N} C_i$

Clustering Coefficient

- **Clustering coefficient:**
  - What portion of i's neighbors are connected?
  - Node i with degree k_i
  - $C_i = \frac{2e_i}{k_i(k_i - 1)}$ where e_i is the number of edges between the neighbors of node i



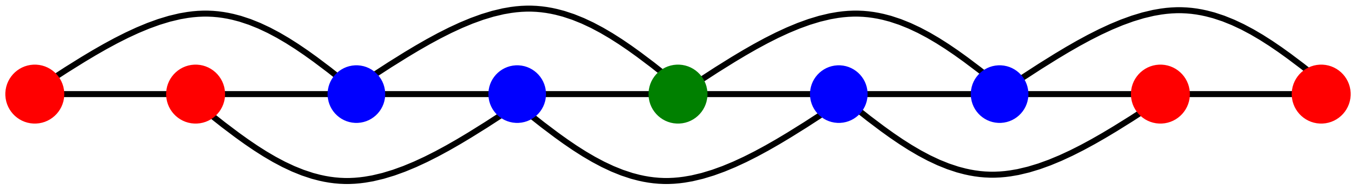
k_B=2, e_B=1, C_B=2/2 = 1

k_D=4, e_D=2, C_D=4/12 = 1/3

Key Network Properties

Degree distribution:	$P(k)$
Path length:	h
Clustering coefficient:	C

Regular Lattice: 1D



- $P(k) = \delta(k\text{-}4)$ $\qquad k=4$ for each node
- $C = ½$ for each node if $N>6$
- Path length:

$$h_{max} \approx \frac{N}{2}$$

Alternative calculation:

$$\sum_{h=1}^{h_{max}} 4 \approx N \quad \Rightarrow \quad h_{max} \approx \frac{N}{4}$$

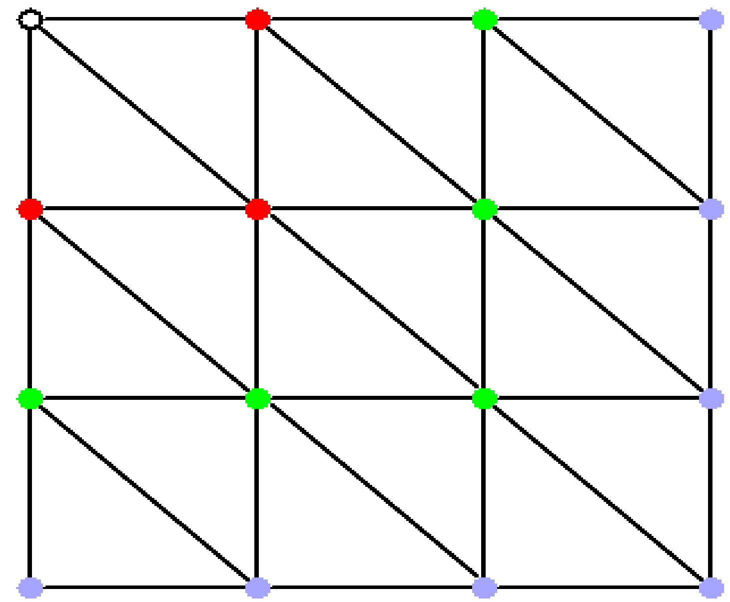
  - The average path-length is $\bar{h} \approx N$
  - Constant degree, constant clustering coefficient.

Regular Lattice: 2D

- $P(k) = \delta(k-6)$
  - $k=6$ for each inside node
- $C = 6/15$ for inside nodes
- Path length:

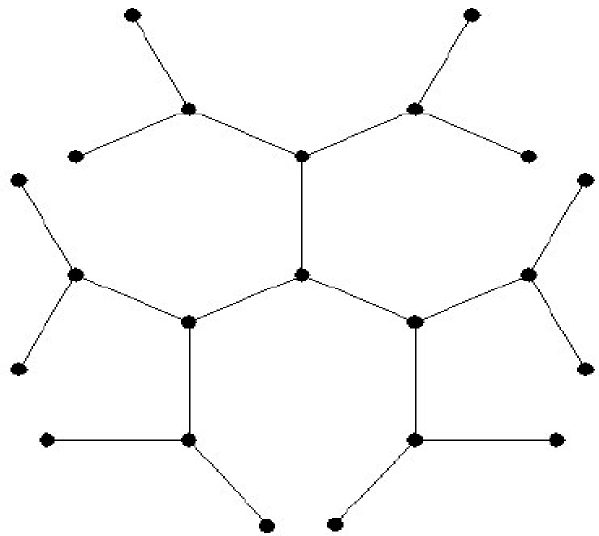
$$\sum_{h=1}^{h_{\max}} 6h \approx N \;\Rightarrow\; \mathrm{h}_{\max} \propto \sqrt{N}$$

  - In general, for lattices:
    - average path-length is $\bar{h} \approx N^{1/D}$
    - Constant degree, constant clustering coefficient



3-way Cayley Tree

- Degree: $\bar{k} = 2$
  - $k=3$ for non-leaves
  - $k=1$ for leaves
- $C = 0$
- Path length:



$$3\sum_{h=1}^{h_{\max}} 2^{h-1} \approx N \;\Rightarrow\; \mathrm{h}_{\max} \propto \log_{\bar{k}} N = \frac{\log N}{\log \bar{k}}$$

$$\int_1^{h_{\max}} 2^{h-1}\,dx = \left.\frac{2^h}{h}\right|_1^{h_{\max}} = \frac{2^{h_{\max}}}{h_{\max}} - 2 \approx 2^{h_{\max}}$$

$$2^{h_{\max}} = N \Rightarrow h_{\max} = \log_2 N$$

  - Distances vary logarithmically with N.

  Constant degree, no clustering.

Erdös-Renyi Random Graph Model

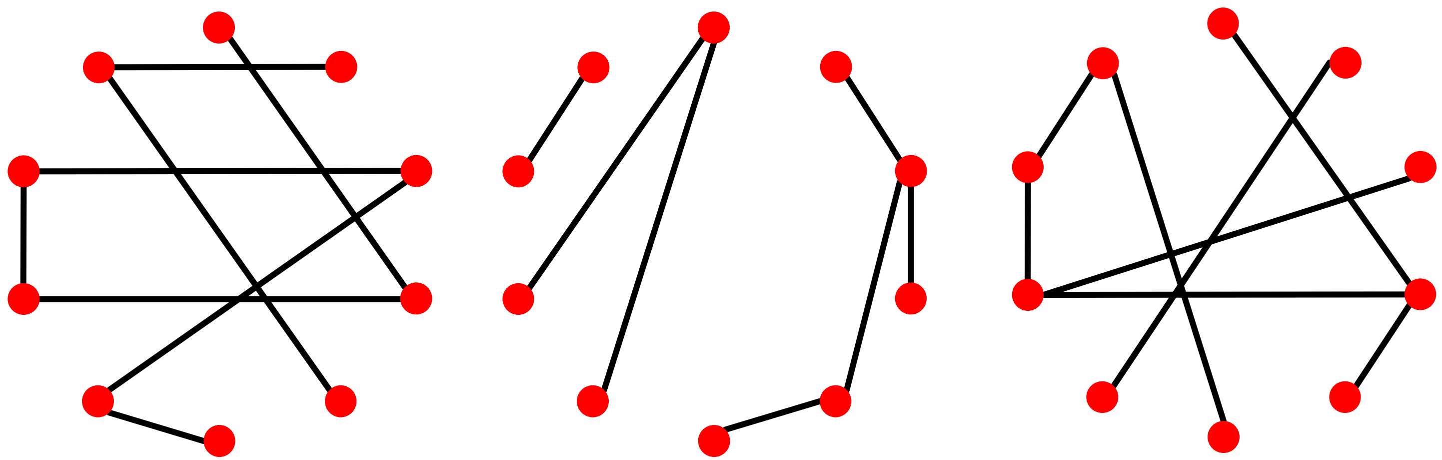
Simplest model of graphs?

- **Erdös-Renyi Random Graph** [Erdös-Renyi, '60]
- Two variants:
  - $G_{n,p}$: undirected graph on n nodes and each edge (u,v) appears i.i.d. with probability p
  - $G_{n,m}$: undirected graph with n nodes, and m uniformly at random picked edges

What kinds of networks does such model produce?

Random Graph Model

- ***n* and *p* do not uniquely define the graph**
- We can have many different realizations. **How many?**



n = 10
p= 1/6

The probability of G_{np} to form a *particular* graph $G(N,E)$ is

$$P(G(N,E)) = p^E (1-p)^{\frac{N(N-1)}{2}-E}$$

That is, each concrete graph **G(N,E)** appears with probability **P(G(N,E))**.

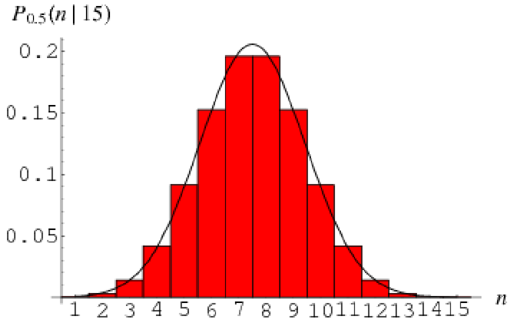
Random Graph Model: Edges

- **How many likely is a graph on E edges?**
- $P(E)$: the probability that a given G_{np} generates a graph on exactly E edges:

$$P(E) = \binom{E_{\max}}{E} p^{E} (1-p)^{E_{\max}-E}$$

where $E_{max}=n(n-1)/2$ is the maximum possible number of edges

Binomial distribution >>>



Node Degrees in a Random Graph

- **What is expected degree of a node?**
- Let X_v be a random var. measuring the degree of the node v: $E[X_v] = \sum_{j=0}^{n-1} j\, P(X_v = j)$
  - Linearity of expectation:
    - For any random variables $Y_1, Y_2, \ldots, Y_k$
    - If $Y = Y_1 + Y_2 + \ldots Y_k$, then $E[Y] = \sum_i E[Y_i]$
- Easier way:
  - Decompose X_v in $X_v = X_{v1} + X_{v2} + \ldots + X_{vn-1}$
    - where X_{vu} is a $\{0,1\}$-random variable which tells if edge (v,u) exists or not

$$E[X_v] = \sum_{y=1}^{n-1} E[X_{vu}] = (n-1)p$$

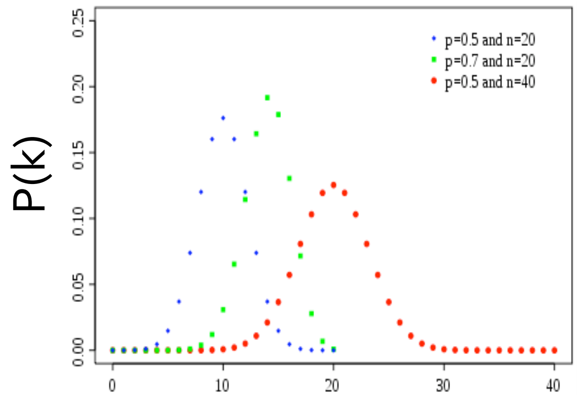
How to think about this?

- Prob. of node u linking to node v is p
- u can link (flips a coin) for all other $(n\text{-}1)$ nodes
- Thus, the expected degree of node u is: $p(n\text{-}1)$

Degree Distribution

- **Degree distribution of G_{np} is Binomial.**
- Let $P(k)$ denote a fraction of nodes with degree k:

$$P(k) = \underbrace{\binom{n-1}{k}}_{\text{Select k nodes from n-1}} \underbrace{p^k}_{\text{Probability of having } k \text{ edges}} \underbrace{(1-p)^{n-1-k}}_{\text{Probability of missing n-1-k edges}}$$



$$\bar{k} = p(n-1)$$

$$\sigma_k^2 = p(1-p)(n-1)$$

$$\frac{\sigma_k}{\bar{k}} = \left[\frac{1-p}{p}\frac{1}{(n-1)}\right]^{1/2} \approx \frac{1}{(n-1)^{1/2}}$$

As the network size increases, the distribution becomes increasingly narrow—we are increasingly confident that the degree of a node is in the vicinity of $\bar{k}$.

Clustering Coefficient of G_{np}

$$C_i = \frac{2e_i}{k_i(k_i-1)}$$

Since edges in G_{np} appear i.i.d with probability p

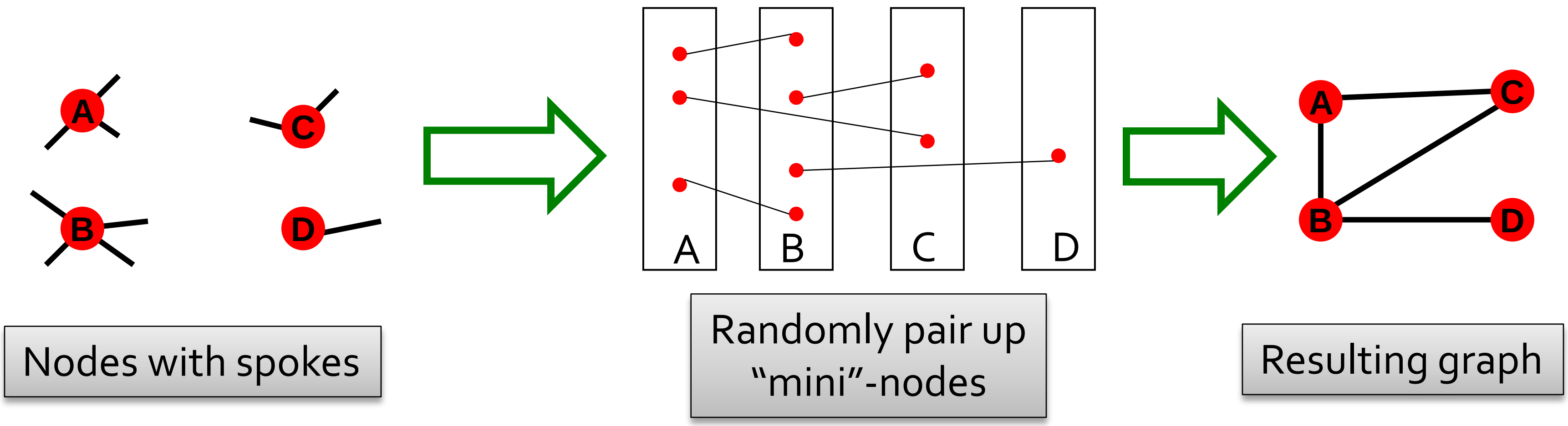
$$e_i \cong p\frac{k_i(k_i-1)}{2} \quad \Rightarrow \quad C \cong p = \frac{\bar{k}}{N}$$

Clustering coefficient of a random graph is small.

For a fixed degree C decreases with the graph size N.

Side-note: Configuration Model

- **Configuration model:**



- Assume a degree sequence $k_1, k_2, \ldots k_N$
- **Useful for as a "null" model of networks**
  - We can compare the real network G and a "random" graph G' which has the same degree sequence as G